K+E SLIDE RULES

DECI-LON®

AN INSTRUCTION MANUAL

KEUFFEL & ESSER CO.

K+E

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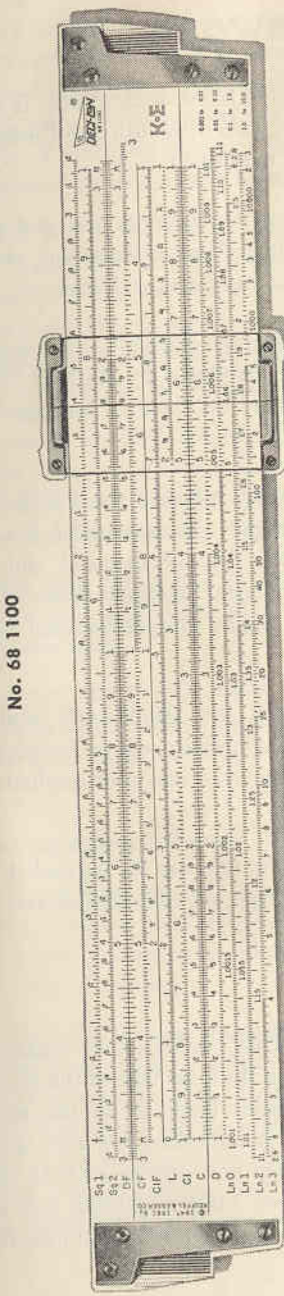
DECI-LON®

No. 68 1100 ENGINEERING—SCIENCE SLIDE RULE



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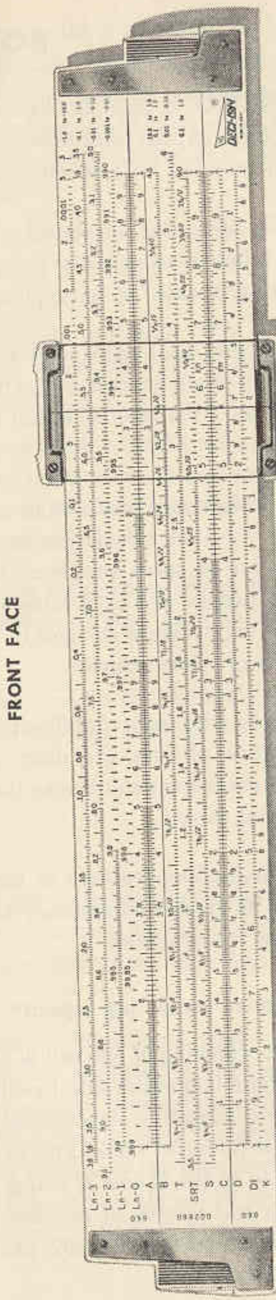

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FOREWORD

The DECI-LON Slide Rule marks the latest achievement in slide rules by K&E, the company that introduced slide rules into the United States and has pioneered in their development ever since.

DECI-LON was developed to provide students and professionals with all the familiar scales of the slide rule for basic calculations, plus new scales and arrangements for more advanced mathematical analysis.

The name DECI-LON combines the prefix "deci-", signifying the decimal division of trigonometric scales first introduced by K&E in 1929, with the suffix "-lon", the pronunciation of the "Ln" symbol for logarithms to the base *e*.

The professional engineer or scientist will find the DECI-LON an extremely versatile calculating instrument. Its expanded Lon scales will enable him to solve problems in the field of financial and investment mathematics, as well as in rates of chemical reactions, decay of radioactive isotopes, temperature changes in jet engine compressors and turbines, and the orbits of space vehicles.

Students who gain familiarity with the DECI-LON while studying mathematics, physics, chemistry or engineering will be better prepared to use its advanced capabilities as they move on into post-graduate studies or professional careers.

The outstanding features of the new rule are its expanded computing capacity, its greater consistency and logic, its convenience and speed, and its lifetime construction.

1. Expanded computing capacity

A slight widening of one rail of the DECI-LON's body makes it possible to locate four Lon scales side-by-side on the front face of the rule, with four Lon minus (negative exponential) scales together on the reverse face. With these powerful scales appearing in unbroken sequence and referring consistently to the *C* and *D* scales which now appear on both faces of the rule, roots and powers of numbers from 1.001 to 30,000, and of decimal fractions from 0.00003 to 0.999, can be found speedily and directly without reversing the rule.

The addition of *Ln0* and *Ln-0* scales brings the lower limit of the Lon scales and the upper limit of the Lon minus scales ten times closer to unity than on previous slide rules. The relationship of these new scales to the *D* scales on both sides of the rule creates, in effect, an infinite series of Lon scales capable of bridging the gap between scale limits and unity to whatever degree of closeness may be required.

The two new Lon scales will be found exceptionally useful in problems of geometric progression involving small rates of change and long periods of time. Such problems include the compounding of interest on a daily basis, the determination of half-lives of isotopes with slow rates of decay, and the estimate of the orbits of satellites acted upon by minute forces such as solar radiation or ion-accelerator engines.

Two new scales, *Sq1* and *Sq2*, constitute a double-length scale used with the full-length *D* scale for fast, accurate evaluation of squares and square roots. Because these new scales adjoin the *DF* scale, the area of a circle can be found instantly when its radius is known.

At the same time, the *A* and *B* scales are retained as scales of calculation, for continuous operations of multiplication and division involving squares or square roots.

Fourth powers and fourth roots, which occur in thermal radiation problems, can be read directly by using the new *Sq1* and *Sq2* scales in conjunction with the *A* and *B* scales.

2. Greater consistency and logic

Slide rule operations are easy to learn and easier still to remember if the rule functions logically—not only in the selection, location and arrangement of the scales, but also in such important details as their color, numbering and direction of reading.

On the DECI-LON, new scales as well as traditional scales maintain the K&E-pioneered principles of full logic and consistency. All scales read directly to the *C* and *D* scales. New scales have been given names which describe their functions—*Sq* for scales that give squares, *Ln* and *Ln-* for scales which give lons (logarithms to the base *e*), *S* for sines and cosines, *T* for tangent and cotangents, and *SRT* for the scale which gives sines, radians, and tangents. The traditional scales *A*, *B*, *C*, *D*, *L*, and *K* remain unchanged.

On DECI-LON, the use of color has been extended to a more consistent level. The two colors, black and red, have these prevailing connotations:

BLACK: Lon scales; positive readings; standard left-to-right reading direction; front face of the rule.

RED: Lon minus scales; negative readings; reverse right-to-left reading direction; reverse face of the rule.

This color consistency is evident in the positioning of the black Lon scales on the front face and the red Lon minus scales on the back face; and in the trigonometric scales, where black is forward reading and slanted to the right, red is reverse reading and slanted to the left.

3. Convenience and speed

A number of features have been incorporated in the DECI-LON to enable the user to perform calculations more rapidly and easily. Among them are: providing C and D scales on both faces; extending the calibrations of the Lon, Square, folded and trigonometric scales beyond the indexes for easier reading of values near the ends; color coding of the legends of scales; and the use of red hairlines which contrast vividly with the black graduation lines of the scales.

4. Lifetime construction

The DECI-LON slide rule is made of a special shatter-proof synthetic material exclusive with K&E. Humidity variations have no effect on its operation. The DECI-LON will not warp or stick. Precision molding and new four-bolt end plates insure accuracy, rigidity and permanence of alignment. Thus DECI-LON, including its unbreakable indicator, is capable of a lifetime of service and is guaranteed to give it.

This manual is designed to enable any interested person to learn to use the slide rule efficiently. The beginner should keep his slide rule before him while reading the manual, should make all settings described in the illustrative examples, and should compute answers for a large number of the exercises. The principles involved are easily understood, but practice is required to build proficiency in using the slide rule easily and accurately.

Those who are already familiar with K&E's Log Log Duplex DECITRIG® slide rule can quickly familiarize themselves with the new features of the DECI-LON by reading only §§25, 26, 27, and 33, covering the *Sq1* and *Sq2* scales, and Chapter VI, on the Lon scales.

A unique feature of this manual is the "visual summary" at the end of each chapter. The manual also contains special articles on solutions of equations frequently encountered in electrical and electronic engineering (§22); spherical triangles (§§56-58), and financial calculations (§71).

Among teachers and students of the slide rule, there has always existed some difference of opinion on the desirability of incorporating slide rule "theory" into the instructions for using a slide rule.

For those who wish the operating rules without the theory, Chapters I through VI of this manual give straightforward instructions on slide rule settings. For those who feel that rules can be better understood and retained if accompanied by an explanation of the underlying principles, Chapter VII explains how and why a slide rule works. This chapter has been written especially for the slide rule user who is not a mathematician or engineer. It starts with an elementary review of exponents and logarithms, and proceeds logically through the explanation of the Lon scales.

CONTENTS

Chapter I

MULTIPLICATION AND DIVISION

Chapter II

THE PROPORTION PRINCIPLE AND RECIPROCAL SCALES

Chapter III

SQUARES AND SQUARE ROOTS; CUBES AND CUBE ROOTS

Chapter IV

TRIGONOMETRIC SCALES

Chapter V

SOLUTIONS OF TRIANGLES

Chapter VI

THE LON SCALES

CHAPTER I

MULTIPLICATION AND DIVISION

1. Introduction. This chapter explains how to multiply and divide, using the basic C and D scales or the folded CF and DF scales. It also covers multiplication and division by π, and combined operations involving a series of multiplications and divisions. For an explanation of the design of these scales, and the basic operating principles of the slide rule, see Chapter VII.

2. Reading the scales.* Everyone has read a ruler in measuring a length. The number of inches is shown by a number appearing on the ruler, then small divisions are counted to get the number of 16ths of an inch in the fractional part of the inch, and finally in close measurement, a fraction of a 16th of an inch may be estimated. We first read a primary length, then a secondary length, and finally estimate a tertiary length. Exactly the same method is used in reading the slide rule. The divisions on the slide rule are not uniform in length, but the same principle applies.

Figure 1 represents, in skeleton form, the fundamental scale of the slide rule, namely the D scale. An examination of this actual

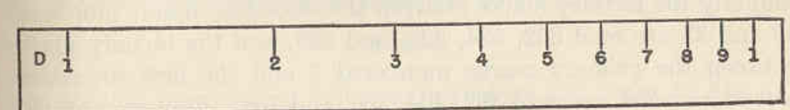

Fig. 1.

scale on the slide rule will show that it is divided into nine parts by primary marks which are numbered 1, 2, 3, . . . , 9, 1. The space between any two primary marks is divided into ten parts by nine secondary marks. These are not numbered on the actual scale except

*The description here given has reference to the 10″ slide rule. However anyone having a rule of different length will be able to understand his rule in the light of the explanation given.

between the primary marks numbered 1 and 2. Fig. 2 shows the secondary marks lying between the primary marks of the *D* scale. Each italicized number drawn on the scale in this illustration (they

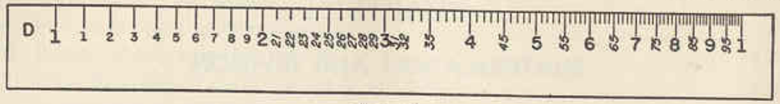

Fig. 2.

do not appear on the actual scale) gives the reading to be associated with its corresponding secondary mark. Thus, the first secondary mark after 2 is numbered 21, the second 22, the third 23, etc.; the first secondary mark after 3 is numbered 31, the second 32, etc. Between the primary marks numbered 1 and 2, the secondary marks are numbered 1, 2, . . . , 9. Evidently the readings associated with these marks are 11, 12, 13, . . . , 19. Finally between the secondary marks, see Fig. 3, appear smaller or tertiary marks which aid in

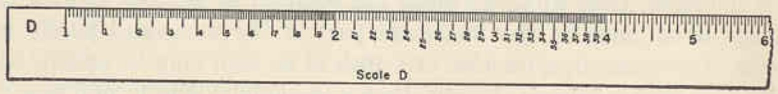

Fig. 3.

obtaining the third digit of a reading. Thus between the secondary marks numbered 22 and 23 there are four tertiary marks. If we think of the end marks as representing 220 and 230, the four tertiary marks divide the interval into five parts each representing 2 units. Hence with these marks we associate the numbers 222, 224, 226, and 228; similarly the tertiary marks between the secondary marks numbered 32 and 33 are read 322, 324, 326, and 328, and the tertiary marks between the primary marks numbered 3 and the first succeeding secondary mark are read 302, 304, 306, and 308. Between any pair of secondary marks to the right of the primary mark numbered 4, there is only one tertiary mark. Hence, each smallest space represents five units. Thus the tertiary mark between the secondary marks representing 41 and 42 is read 415, that between the secondary marks representing 55 and 56 is read 555, and the first tertiary mark to the right of the primary mark numbered 4 is read 405.

The reading of any position between a pair of successive tertiary marks must be based on an estimate. Thus a position half way between the tertiary marks associated with 222 and 224 is read 223, and a position two fifths of the way from the tertiary mark representing

415 to the next mark is read 417. The principle illustrated by these readings applies in all cases.

Consider the process of finding on the *D* scale the position representing 246. The first figure on the left, namely 2, tells us that the position lies between the primary marks numbered 2 and 3. This region is indicated by the brace in Fig. (*a*). The second figure from the left, namely 4, tells us that the position lies between

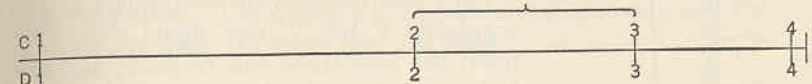

Fig. *a*.

the secondary marks associated with 24 and 25. This region is indicated by the brace in Fig. (*b*). Now there are four marks between

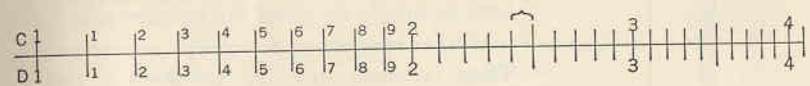

Fig. *b*.

the secondary marks associated with 24 and 25. With these are associated the numbers 242, 244, 246, and 248 respectively. Thus

Fig. *c*.

the position representing 246 is indicated by the arrow in Fig. (*c*). Fig. (*abc*) gives a condensed summary of the process.

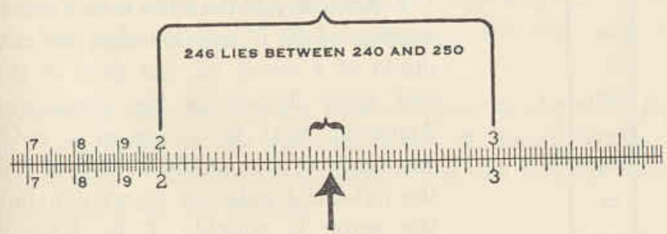

Fig. *abc*.

Fig. 4.

It is important to note that the decimal point has no bearing upon the position associated with a number on the C and D scales. Consequently, the arrow in Fig. (*abc*) may represent 246, 2.46, 0.000246, 24,600, or any other number whose principal digits are 2, 4, 6. The placing of the decimal point will be explained later in this chapter.

For a position between the primary marks numbered 1 and 2, four digits should be read; the first three will be exact and the last one estimated. No attempt should be made to read more than three digits for positions to the right of the primary mark numbered 4.

While making a reading, the operator should have definitely in mind the number associated with the smallest space under consideration. Thus between primary numbers 1 and 2, the smallest division is associated with 10 in the fourth place; between 2 and 4, the smallest division has a value of 2 in the third place; while to the right of 4, the smallest division has a value of 5 in the third place.

The operator should read from Fig. 4 the numbers associated with the marks lettered A, B, C, . . . and compare his readings with the following numbers: A 365, B 327, C 263, D 1750, E 1347, F 305, G 207, H 1075, I 435, J 427.

3. Accuracy of the slide rule. From the discussion of §2 it appears that we read four digits of a result on one part of the scale and three figures on the remaining part. Assuming that the error of a reading is one tenth of the smallest interval following the left-hand index of D, we conclude that the error is roughly 1 in 1000 or one tenth of one per cent. The effect of the

assumed error in judging a distance is inversely proportional to the length of the rule. Hence we associate with a 10-inch slide rule an error of one tenth of one per cent, with a 20-inch slide rule an error of one twentieth of one per cent or 1 part in 2000, and with the Thacher Cylindrical slide rule an error of a hundredth of one per cent or one part in 10,000. The accuracy obtainable with the 10-inch slide rule is sufficient for many practical purposes; in any case the slide rule serves as a check.

4. Definitions. The middle sliding part of the slide rule (see Fig. 5) is called the **slide,** the other part the **body.**

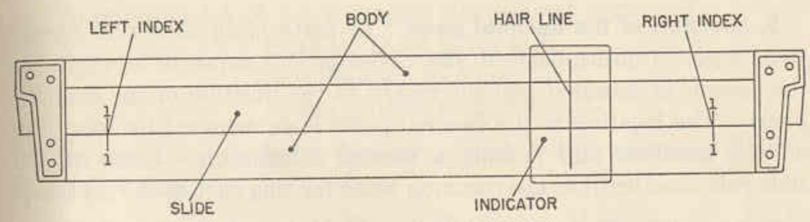

Fig. 5.

The transparent runner is referred to as the **indicator,** and the line on it is called the **hairline.**

The mark opposite the primary number 1 on the D and C scale is called the **index** of the scale. The C and D scales have two indexes, one at the left end called the **left index,** the other at the right end called the **right index.**

A number on one scale is said to be **opposite** a number on another scale if the hairline can cover both numbers at the same time. Each number is said to be **opposite** the other.

The slide rule is said to be **closed** when the slide is in such a position that the left index of the C scale is opposite the left index of the D scale.

Mathematical calculations are accomplished on the slide rule by moving the hairline or the slide or both. A description of these movements and of the resulting positions of the hairline and slide will be referred to as the **setting.**

Many settings will be described throughout this manual. In these descriptions two expressions, **"push the hairline"** and **"draw the number,"** will appear frequently. These two phrases are virtually idiomatic in slide rule language.

The meaning of the first phrase, "push the hairline," is obvious. The phrase, "draw the number," is used to describe the operation of moving the slide to bring a number on the slide into a new position relative to the body. Therefore the word "draw," when used in these settings, should always be associated with movement of the slide.

Such words as "close" and "opposite" will be used repeatedly in this manual. Moreover, the abbreviation *C* will be used for *C scale*, *D* for *D scale*, etc.

5. Location of the decimal point. In performing slide rule operations such as multiplication and division, the sequence of digits in the answer is obtained without regard to the position of the decimal point. The location of the decimal point is determined by rounding off the numbers and making a mental calculation. Users of the slide rule soon learn to use common sense for this part of the problem.

For example, if the slide rule is used to multiply 16.75 by 2.83, the three digits of the answer produced by the rule will be 474. To place the decimal point it could be noted that the answer is approximately $16 \times 3 = 48$. Thus the answer is obviously **47.4.**

6. Multiplication. The process of multiplication may be performed by using scales *C* and *D*. The *C* scale is on the slide, but in other respects it is like the *D* scale and is read in the same manner.

To multiply 2 by 4 (Fig. 6),

to 2 on *D* set index of *C*,
push hairline to 4 on *C*,
at the hairline read **8** on *D*.

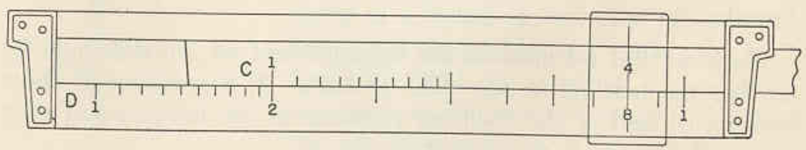

Fig. 6.

To multiply 3×3 (Fig. 7),

to 3 on *D* set index of *C*,
push hairline to 3 on *C*,
at the hairline read **9** on *D*.

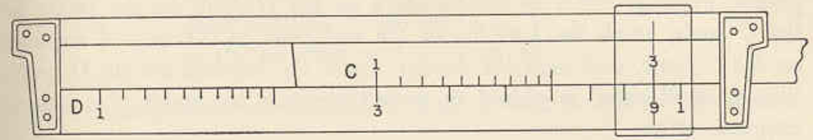

Fig. 7.

To multiply 1.5×3.5, disregard the decimal point and

to 15 on *D* set index of *C*,
push hairline to 35 on *C*,
at the hairline read 525 on *D*.

By inspection we know that the answer is approximately 5 and is therefore **5.25.**

To find the value of 16.75×2.83 (Fig. 8), disregard the decimal point and

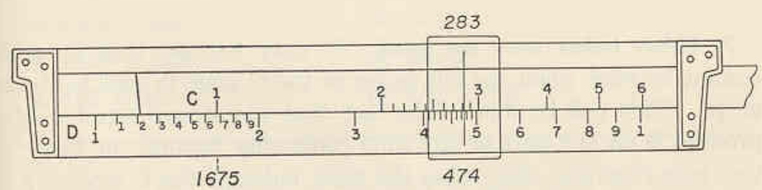

Fig. 8.

to 1675 on *D* set index of *C*,
push hairline to 283 on *C*,
at the hairline read 474 on *D*.

To place the decimal point we approximate the answer by noting that it is approximately $3 \times 16 = 48$. Hence the answer is **47.4.**

To find the value of 0.001753×12.17,

to 1753 on *D* set left index of *C*,
push hairline to 1217 on *C*,
at the hairline read 2133 on *D*.

To place the decimal point, approximate the answer by writing $.002 \times 10 = .02$. Hence the answer is **0.02133**.

These examples illustrate the use of the following rule:

Rule. *To find the product of two numbers, disregard the decimal points, opposite either of the numbers on the D scale set the index of the C scale, push the hairline of the indicator to the second number on the C scale, and read the answer under the hairline on the D scale. The decimal point is placed in accordance with the result of a mental approximation.*

EXERCISES

1. 3×2.	6. 1.75×5.5.	11. 1.047×3080.
2. 3.5×2.	7. 4.33×11.5.	12. 0.00205×408.
3. 5×2.	8. 2.03×167.3.	13. $(3.142)^2$.
4. 2×4.55.	9. 1.536×30.6.	14. $(1.756)^2$.
5. 4.5×1.5.	10. 0.0756×1.093.	

7. Either index may be used. It may happen that a product cannot be read when the left index of the *C* scale is used in the rule of §6. This will be due to the fact that the second number of the product is on the part of the slide projecting beyond the body. In this case reset the slide using the right index of the *C* scale in place of the left, or use the following rule:

Rule. *When a number is to be read on the D scale opposite a number of the C scale and cannot be read, push the hairline to the index of the C scale inside the body and draw the other index of the C scale under the hairline. Then make the desired reading. This operation is called* ***"interchanging the indexes."***

This rule, slightly modified to apply to the scales being used, is generally applicable when an operation calls for setting the hairline to a position on the part of the slide extending beyond the body.

If, to find the product of 2 and 6, we set the left index of the

C scale opposite 2 on the *D* scale, we cannot read the answer on the *D* scale opposite 6 on the *C* scale. Hence, we set the right index of *C* opposite 2 on *D*; opposite 6 on *C* read the answer, **12**, on *D*.

Again, to find 0.0314×564,
to 314 on *D* set the right index of *C*,
push hairline to 564 on *C*,
at the hairline read 1771 on *D*.

An approximation is obtained by finding $0.03 \times 600 = 18$. Hence the product is **17.71**.

EXERCISES

Perform the indicated multiplications:

1. 3×5.	**9.** 912×0.267.
2. 3.05×5.17.	**10.** 48.7×1.173.
3. 5.56×634.	**11.** 0.298×0.544.
4. 743×0.0567.	**12.** 0.0456×4.40.
5. 0.0495×0.0267.	**13.** 8640×0.01973.
6. 1.876×926.	**14.** $(75.0)^2$.
7. 1.876×5.32.	**15.** $(83.0)^2$.
8. 42.3×31.7.	**16.** 4.98×576.

8. Division. The process of division is performed by using the *C* and *D* scales.

To divide 8 by 4 (Fig. 9),
push hairline to 8 on *D*,
draw 4 of *C* under the hairline,
opposite index of *C* read 2 on *D*.

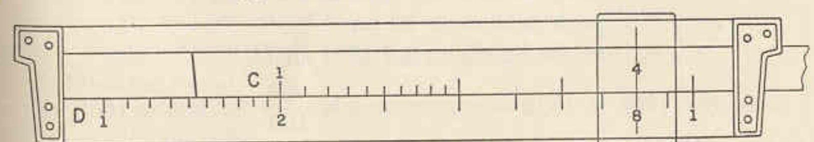

FIG. 9.

To divide 876 by 20.4,

push hairline to 876 on D,
draw 204 of C under the hairline,
opposite index of C read 429 on D.

The mental calculation $800 \div 20 = 40$ shows that the decimal point must be placed after the 2. Hence the answer is **42.9.**

These examples illustrate the use of the following rule:

Rule. *To find the quotient of two numbers, disregard the decimal points, opposite the numerator on the D scale set the denominator on the C scale, opposite the index of the C scale read the quotient on the D scale. The position of the decimal point is determined from information gained by making a mental calculation.*

EXERCISES

Perform the indicated operations:

1. $87.5 \div 37.7$.
2. $3.75 \div 0.0227$.
3. $0.685 \div 8.93$.
4. $1029 \div 9.70$.
5. $0.00377 \div 5.29$.
6. $2875 \div 37.1$.
7. $871 \div 0.468$.
8. $0.0385 \div 0.001462$.
9. $3.14 \div 2.72$.
10. $3.42 \div 81.7$.
11. $529 \div 565$.
12. $0.0456 \div 0.0297$.
13. $396 \div 0.643$.
14. $0.0592 \div 1.983$.
15. $0.378 \div 0.0762$.
16. $10.05 \div 30.3$.

9. Simple applications, percentage, rates. Many problems involving percentage and rates are easily solved by means of the slide rule.

One per cent (1%) of a number N is $N \times 1/100$; hence 5% of N is $N \times 5/100$, and, in general, p% of N is $pN/100$. Hence to find 83% of 1872

to 1872 on D set right index of C,
push hairline to 83 on C,
at the hairline read 1554 on D.

Since $(83/100) \times 1872$ is approximately $\frac{80}{100} \times 2000 = 1600$, the answer is **1554.**

To find the answer to the question "M is what per cent of N?" we

must find $100\ M \div N$. Thus, to find the answer to the question "87 is what per cent of 184.7?" we must divide $87 \times 100 = 8700$ by 184.7. Hence

push hairline to 87 on D,
draw 1847 of C under the hairline,
opposite index of C read 471 on D.

The mental calculation $\frac{9000}{200} = 45$ shows that the decimal point should be placed after the 7. Hence the answer is **47.1%.**

For a body moving with a constant velocity, distance = rate times time. Hence if we write d for distance, r for rate, and t for time, we have

$$d = rt, \text{ or } r = \frac{d}{t}, \text{ or } t = \frac{d}{r}.$$

To find the distance traveled by a car going 33.7 miles per hour for 7.75 hours, write $d = 33.7 \times 7.75$, and

to 337 on D set right index of C,
push hairline to 775 on C,
at hairline read 261 on D.

Since the answer is nearly $8 \times 30 = 240$ miles, we have $d =$ **261** miles.

To find the average rate at which a driver must travel to cover 287 miles in 8.75 hours, write $r = 287 \div 8.75$, and

push hairline to 287 on D,
draw 875 of C under the hairline,
opposite the index of C read 328 on D.

Since the rate is near $280 \div 10 = 28$, we have $r =$ **32.8** miles per hour

EXERCISES

1. Find (*a*) 86.3 per cent of 1826.
   (*b*) 75.2 per cent of 3.46.
   (*c*) 18.3 per cent of 28.7.
   (*d*) 0.95 per cent of 483.
2. What per cent of
   (*a*) 69 is 18?
   (*b*) 132 is 85?
   (*c*) 87.6 is 192.8?
   (*d*) 1027 is 28?

3. Find the distance covered by a body moving
   (*a*) 23.7 miles per hour for 7.55 hours.
   (*b*) 68.3 miles per hour for 1.773 hours.
   (*c*) 128.7 miles per hour for 16.65 hours.
4. At what rate must a body move to cover
   (*a*) 100 yards in 10.85 seconds?
   (*b*) 386 feet in 25.7 seconds?
   (*c*) 93,000,000 miles in 8 minutes and 20 seconds?
5. Find the time required to move
   (*a*) 100 yards at 9.87 yards per second.
   (*b*) 3800 miles at 128.7 miles per hour.
   (*c*) 25,000 miles at 77.5 miles per hour.

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10. Use of the scales *DF* and *CF* (folded scales). The *DF* and the *CF* scales are the same as the *D* and the *C* scales respectively except in the position of their indexes. The fundamental fact concerning the folded scales may be stated as follows: *if for any setting of the slide, a number M of the C scale is opposite a number N on the D scale, then the number M of the CF scale is opposite the number N on the DF scale.* Thus, if the operator will draw 1 of the *CF* scale opposite 1.5 on the *DF* scale, he will find the following opposites on the *CF* and *DF* scales

DF	1.5	3	6	7.5	9	1
CF	1	2	4	5	6	6.67

and the same opposites will appear on the *C* and *D* scales.

The following statement relating to the folded scales is basic. *The process of* **setting the hairline** *to a number N on scale C to find its opposite M on scale D may be replaced by* **setting the hairline** *to N on scale CF to find its opposite M on scale DF.* The statement holds true if letters *C* and *D* are interchanged.

In accordance with the principle stated above, if the operator wishes to read a number on the *D* scale opposite a number *N* on the *C* scale but cannot do so, he can generally read the required number on the *DF* scale opposite *N* on the *CF* scale. For example to find 2×6,

to 2 on *D* set left index of *C*,
push hairline to 6 on *CF*,
at the hairline read 12 on *DF*.

By using the *CF* and *DF* scales we saved the trouble of moving the slide as well as the attendant source of error. This saving, entering as it does in many ways, is the main reason for using the folded scales.

The folded scales may be used to perform multiplications and divisions just as the *C* and *D* scales are used. Thus to find 6.17×7.34,

to 617 on *DF* set index of *CF*,
push hairline to 734 on *CF*,
at the hairline read **45.3** on *DF*; or
to 617 on *DF* set index of *CF*,
push hairline to 734 on *C*,
at the hairline read **45.3** on *D*.

Again to find the quotient 7.68/8.43,

push hairline to 768 on *DF*,
draw 843 of *CF* under the hairline,
opposite the index of *CF* read **0.911** on *DF*; or
push hairline to 768 on *DF*,
draw 843 of *CF* under the hairline,
opposite the index of *C* read **0.911** on *D*.

It now appears that we may perform a multiplication or a division in several ways by using two or more of the scales *C*, *D*, *CF*, and *DF*. The sentence written in italics near the beginning of the article sets forth the guiding principle.

EXERCISES

Perform each of the operations indicated in the following exercises. Whenever possible without resetting, read the answer on *D* and also on *DF*:

1. 5.78×6.35.
2. 7.84×1.065.
3. $0.00465 \div 73.6$.
4. $0.0634 \times 53{,}600$.
5. $1.769 \div 496$.
6. $946 \div 0.0677$.
7. 813×1.951.
8. $0.00755 \div 0.338$.
9. $0.0948 \div 7.23$.
10. $149.0 \div 63.3$.
11. $2.718 \div 65.7$.
12. $1.072 \div 10.97$.

---

11. Multiplication and division by π using scales *CF* and *DF*.* The symbol π (pronounced pi) is used to designate the ratio of the circumference of a circle to its diameter. The value of π accurate to four decimal places is 3.1416.

*The *C* and *D* scales on the reverse ("red") face of the slide rule have locating marks for convenience in multiplying or dividing by π, 2π, or $\frac{\pi}{4}$. See Appendix A for an explanation.

By means of the *CF* and *DF* scales operating with scales *C* and *D*, multiplication and division by π is accomplished with only a single setting of the hairline. This is made possible because the index of the *CF* and *DF* scales is so placed in relation to the indexes of the *C* and *D* scales respectively that:

any number on DF is π times its opposite on D

or

any number on D is $\frac{1}{\pi}$ times its opposite on DF.

The statement in italics applies to the *C* and *CF* scales also.

Thus to find the value of 5π,

push hairline to 5 on *D*,
under hairline read **15.7** on *DF*.

To find the value of $\frac{4}{\pi}$,

push hairline to 4 on *DF*,
under hairline read **1.273** on *D*.

Example. The circumference of a circle measures 8.48 inches. Find its diameter.

Solution. The formula for the circumference (C) of a circle in terms of its diameter (d) is

$$C = \pi d \quad \text{or} \quad d = \frac{C}{\pi}.$$

Therefore, $d = \frac{8.48}{\pi}$. To find d we make the following setting:

push hairline to 848 on *DF*,
under hairline read **2.70** on *D*.

The position of the decimal point was determined by the approximation, $\frac{9}{3} = 3$. Therefore the diameter d = **2.70** inches.

EXERCISES

Find the value of the following:

1. 6π.
2. 8.4π.
3. 78.3π.
4. $\frac{6}{\pi}$.
5. $\frac{78.3}{\pi}$.
6. 0.504π.
7. 0.0876π.
8. $15/6\pi$.
9. $\frac{19.6}{\pi}$.
10. $\frac{17.14}{\pi}$.
11. The diameter of a circle is 2.84 inches. Find its circumference.

12. The circumference of a circle is 19.63 inches. Find its diameter.
13. A cylindrical tube is 13 inches long and has an outside diameter of $2\frac{1}{8}$ inches. Find its outside surface area. The formula for the outside surface area is $S = \pi dh$ where S denotes the surface area, d the diameter and h the length.

12. Combined multiplication and division.

Example 1. Find the value of $\frac{7.36 \times 8.44}{92}$.

Solution. Reason as follows: first divide 736 by 92 and then multiply the result by 844. This would suggest that we

push hairline to 736 on *D*,
draw 92 of *C* under the hairline,
opposite 844 on *C*, read **0.675** on *D*.

Example 2. Find the value of $\frac{18 \times 45 \times 37}{23 \times 29}$.

Solution. Reason as follows: (*a*) divide 18 by 23, (*b*) multiply the result by 45, (*c*) divide this second result by 29, (*d*) multiply this third result by 37. This argument suggests that we

push hairline to 18 on *D*,
draw 23 of *C* under the hairline,
push hairline to 45 on *C*,
draw 29 of *C* under the hairline,
push hairline to 37 on *C*,
at the hairline read 449 on *D*.

To determine the position of the decimal point write $\frac{20 \times 40 \times 40}{20 \times 30}$ = about 50. Hence the answer is **44.9.**

A little reflection on the procedure of Example 2 will enable the operator to evaluate by the shortest method expressions similar to the one just considered. He should observe that: the *D* scale was used only twice, once at the beginning of the process and once at its end; *the process for each number of the denominator consisted in drawing that number, located on the C scale, under the hairline; the process for each number of the numerator consisted in pushing the hairline to that number located on the C scale.*

If at any time the indicator cannot be placed because of the projection of the slide, interchange the indexes or carry on the operations using the folded scales.

EXERCISES

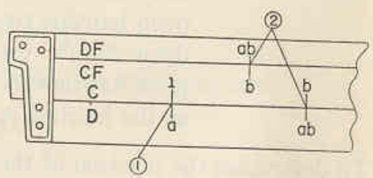
1. $\frac{9 \times 14}{5}$.
2. $\frac{37.4 \times 5.96\,\pi}{75.6}$.
3. $\frac{146.2 \times 8.50}{3290\,\pi}$.
4. $\frac{11 \times 12 \times 27\pi}{7 \times 13}$.
5. $\frac{65.6 \times 0.842}{4.63}$.
6. $\frac{76.6 \times 63.4 \times 96}{3.23\,\pi}$.
7. $\frac{47.2 \times 18.3\,\pi}{32.6 \times 16.4}$.
8. $\frac{3.82 \times 6.94 \times 7.82 \times 426}{77.8 \times 0.0322 \times 642}$.
9. Multiply 312 successively by 1.44, 2.62, 3.18, 4.6, 5.12, 6.72, 7.46, 8.12, 9.62.

Hint: draw the left index of C to 312 on D, push hairline in succession to the given numbers on C or CF and read the answers under the hairline on D or DF respectively.

13. Visual summary.*

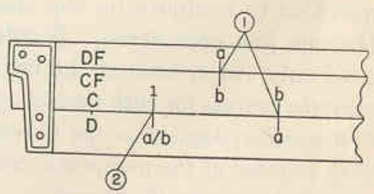
To multiply a by b: $x = a \times b$.

1. To a on D set either index of C,
2. at b on C (CF) read ab on D (DF).



To divide a by b: $x = \frac{a}{b}$.

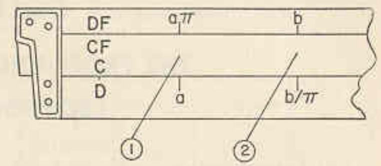
1. To a on D (DF) set b on C (CF),
2. at index of C read a/b on D.



* In these visual summaries, lower case letters from beginning of alphabet – a, b, c, etc. – represent known quantities; letters from end of alphabet – x, y, z, – represent unknown quantities to be found. Capital letters – A, B, C, etc., – designate scales.

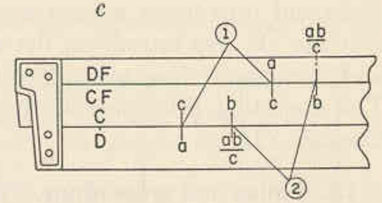
To multiply or divide by π: $x = \pi a$; $x = \frac{b}{\pi}$.

1. At a on D read πa on DF;
2. at b on DF read $\frac{b}{\pi}$ on D.



Combined multiplication and division: $x = \frac{ab}{c}$.

1. To a on D (DF) set c on C (CF),
2. at b on C (CF) read $\frac{ab}{c}$ on D (DF).



CHAPTER II

THE PROPORTION PRINCIPLE AND RECIPROCAL SCALES

14. Introduction. This chapter introduces the important concept of the proportional relationship between numbers on the C and D scales, and shows how this relationship can be used for solving equations and converting measurements into their equivalents in other systems. It also introduces the reciprocal scales CI, DI, and CIF, and shows how these scales facilitate various types of computations. The operating principles of the reciprocal scales are explained in Chapter VII.

15. Ratios and proportions. The *ratio* of two numbers a and b is the quotient of a divided by b or a/b. A statement of equality between two ratios is called a *proportion*. Thus

$$\frac{2}{3} = \frac{6}{9}, \quad \frac{x}{5} = \frac{7}{11}, \quad \frac{a}{b} = \frac{c}{d}$$

are proportions. We shall at times refer to equations having such forms as

$$\frac{2}{3} = \frac{x}{5} = \frac{9}{y} = \frac{10}{z}, \text{ and } \frac{a}{b} = \frac{c}{d} = \frac{e}{f}$$

as proportions.

An important setting like the one for multiplication, the one for division, and any other one that the operator will use frequently, should be practiced until it is made without thought. But, in the process of devising the best settings to obtain a particular result, of making a setting used infrequently, or of recalling a forgotten setting, the application of proportions as explained in the next article is very useful.

16. Use of proportions. If the slide is drawn to any position, the ratio of any number on the D scale to its opposite on the C scale is, in accordance with the setting for division, equal to the number on

the D scale opposite the index on the C scale. In other words, *when the slide is set in any position, the ratio of any number on the D scale to its opposite on the C scale is the same as the ratio of any other number on the D scale to its opposite on the C scale.* For example draw 1 of C

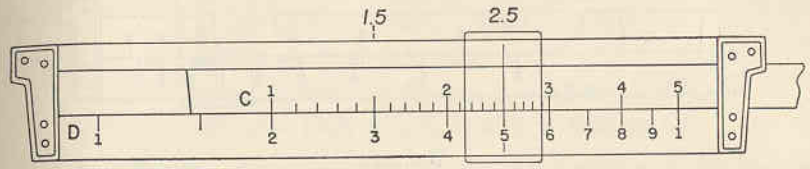


Fig. 10.

opposite 2 on D (see Fig. 10) and find the opposites indicated in the following table:

C (or CF)	1	1.5	2.5	3	4	5
D (or DF)	2	3	5	6	8	10

and draw 2 of C over 1 on D and read the same opposites. The same statement is true if in it we replace C scale by CF scale and D scale by DF scale. Hence, *if both numerator n and denominator d of a ratio in a given proportion are known, we can set n of the C scale opposite d on the D scale and then read, for an equal ratio having one part known, its unknown part opposite the known part.* We could also begin by setting d on the C scale opposite n on the D scale. It is important to observe that *all the numerators of a series of equal ratios must appear on one scale and the denominators on the other.* For example, let it be required to find the value of x satisfying

$$\frac{x}{56} = \frac{9}{7}.$$

Here the known ratio is $9/7$. Hence

push hairline to 7 on D,
draw 9 of C under the hairline,
push hairline to 56 on D,
at the hairline read 72 on C; or
push hairline to 9 on D,
draw 7 of C under the hairline,
push hairline to 56 on C,
at the hairline read 72 on D.

Figure 11 indicates the setting. The *CF* and *DF* scales could have been used to obtain exactly the same settings and results.

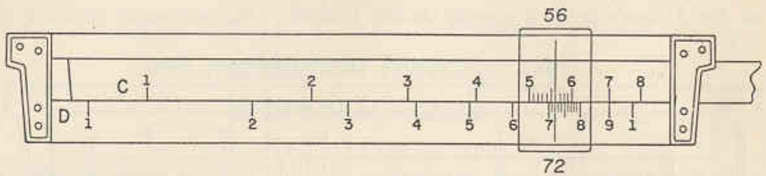


Fig. 11.

To find the values of x, y, and z defined by the equations

$$\frac{C}{D}: \quad \frac{3.15}{5.29} = \frac{x}{4.35} = \frac{57.6}{y} = \frac{z}{183.4},$$

note that C and D indicate the respective scales for the numerators and the denominators, observe that 3.15/5.29 is the known ratio, and

push hairline to 529 on D,
draw 315 of C under the hairline,
opposite 435 on D, read x = **2.59** on C,
opposite 576 on C, read y = **96.7** on D,
opposite 1834 on D, read z = **109.2** on C.

The positions of the decimal points were determined by noticing that each denominator had to be somewhat less than twice its associated numerator because 5.29 is somewhat less than twice 3.15.

When an answer cannot be read, interchange the indexes. Thus to find the values of x and y satisfying

$$\frac{C}{D}: \quad \frac{x}{587} = \frac{14.56}{97.6} = \frac{5.78}{y},$$

to 976 on D set 1456 of C; then, since the answers cannot be read, interchange the indexes, push the hairline to the index on C, draw the right index of C under the hairline and

opposite 587 on D, read x = **87.6** on C,
opposite 578 on C, read y = **38.7** on D.

Here the positions of the decimal points were determined by observing that each denominator had to be about six times the associated numerator.

When a result cannot be read on the C scale nor on the D scale, it may be possible to read it on the CF scale or on the DF scale. Thus, to find x and y satisfying the equations

$$\frac{C \text{ (or } CF)}{D \text{ (or } DF)}: \quad \frac{4.92}{x} = \frac{1}{3.23} = \frac{y}{13.08},$$

to 323 on D set left index of C,
opposite 492 on CF, read x = **15.89** on DF,
opposite 1308 on DF, read y = **4.05** on CF.

If the difference of the first digits of the two numbers of the known ratio is small, use the C and D scales for the initial setting; if the difference is large, use the CF and DF scales. Since in the next to the last example, the difference between the first digits was great, the CF and DF scales should have been used for the initial setting. This would have eliminated the necessity for shifting the slide.

EXERCISES

Find, in each of the following equations, the values of the unknowns:

1. $\frac{x}{5} = \frac{78}{9}$.
2. $\frac{x}{120} = \frac{240}{170}$.
3. $\frac{7}{8} = \frac{249}{x}$.
4. $\frac{2}{3} = \frac{x}{7.83}$.
5. $\frac{x}{1.804} = \frac{y}{25} = \frac{1}{0.785}$.
6. $\frac{x}{709} = \frac{246}{y} = \frac{28}{384}$.
7. $\frac{17}{x} = \frac{1.365}{8.53} = \frac{4.86}{y}$.
8. $\frac{8.51}{1.5} = \frac{9}{x} = \frac{235}{y}$.
9. $\frac{x}{2.07} = \frac{3}{61.3} = \frac{z}{1.571}$.
10. $\frac{x}{0.204} = \frac{y}{0.0506} = \frac{5.28}{z} = \frac{2.01}{0.1034}$.
11. $\frac{0.813}{2.85} = \frac{x}{4.61} = \frac{0.435}{y}$.
12. $\frac{x}{0.429} = \frac{y}{0.789} = \frac{2.43}{0.0276}$.
13. $\frac{x}{0.00560} = \frac{0.743}{1} = \frac{0.0615}{y}$.
14. $\frac{x}{y} = \frac{y}{7.34} = \frac{3.75}{29.7}$.
15. $\frac{x}{49.6} = \frac{z}{y} = \frac{y}{3.58} = \frac{1.076}{0.287}$.

17. Forming proportions from equations. Since proportions are algebraic equations, they may be rearranged in accordance with the laws of algebra. For example, if

$$x = \frac{ab}{c}, \tag{1}$$

we may write the proportion

$$\frac{x}{1} = \frac{ab}{c}, \tag{2}$$

or we may divide both sides by a to get

$$\frac{x}{a} = \frac{ab}{ac}, \quad \text{or} \quad \frac{x}{a} = \frac{b}{c}, \tag{3}$$

or we may multiply both sides by c/x to obtain

$$\frac{cx}{x} = \frac{cab}{xc}, \quad \text{or} \quad \frac{c}{1} = \frac{ab}{x}. \tag{4}$$

Rule (*A*). *A number may be divided by 1 to form a ratio.* This was done in obtaining proportion (2).

Rule (*B*). *A factor of the numerator of either ratio of a proportion may be replaced by 1 and written as a factor of the denominator of the other ratio, and a factor of the denominator of either ratio may be replaced by 1 and written as a factor of the numerator of the other ratio.* Thus (3) could have been obtained from (1) by transferring a from the numerator of the right hand ratio to the denominator of the left hand ratio.

For example, to find $\dfrac{16 \times 28}{35}$, write $x = \dfrac{16 \times 28}{35}$, apply Rule (*B*) to obtain $\dfrac{C}{D}$: $\quad \dfrac{x}{16} = \dfrac{28}{35}$, and

push hairline to 35 on D,
draw 28 of C under the hairline,
opposite 16 on D, read x = **12.8** on C.

Figure 12 indicates the setting.

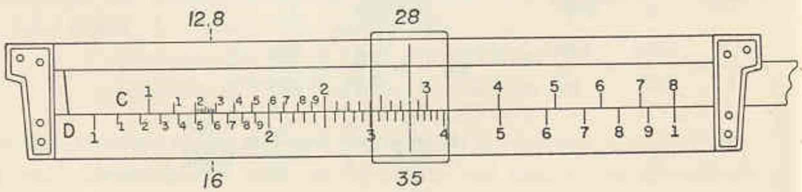


FIG. 12.

To recall the rule for dividing a given number M by a second given number N, write $x = \dfrac{M}{N}$, apply Rule (*A*) to obtain $\dfrac{D}{C}$: $\quad \dfrac{x}{1} = \dfrac{M}{N}$,

and push hairline to M on D,
draw N of C under the hairline,
opposite index of C, read x on D.

To recall the rule for multiplication, set $x = \dfrac{MN}{1}$, apply Rule (*B*) to obtain $\dfrac{D}{C}$: $\quad \dfrac{x}{M} = \dfrac{N}{1}$, and

to N on D set index of C,
opposite M on C, read x on D.

To find x if $\dfrac{1}{x} = \dfrac{864}{(7.48)\ (25.5)}$, use Rule (*B*) to get $\dfrac{7.48}{x} = \dfrac{864}{25.5}$, make the corresponding setting and read x = **0.221.** The position of the decimal point was determined by observing that x must be about $\dfrac{1}{40}$ of 8, or 0.2.

EXERCISES

Find in each case the value of the unknown quantity:

1. $y = \dfrac{8 \times 12}{7}$.
2. $7.4 = \dfrac{9y}{28}$.
3. $8y = 75.6 \times 9$.
4. $y = \dfrac{86 \times 70.8}{125}$.
5. $y = \dfrac{147.5 \times 8.76}{3260}$.
6. $y = \dfrac{0.797 \times 5.96}{0.502}$.
7. $\dfrac{37 \times 86}{y} = 75.7$.
8. $498 = \dfrac{89.3x}{0.563}$.
9. $0.874 = \dfrac{3.95 \times 0.707}{x}$.
10. $0.695 = \dfrac{0.0879}{x}$.
11. $\dfrac{1}{386} = \dfrac{0.772}{2.85y}$.
12. $2580y = 17.9 \times 587$.
13. $3.14y = 0.785 \times 38.7$.
14. $\dfrac{0.876y}{5.49} = 7.59$.

18. Equivalent expressions of quantity.* When the value of a quantity is known in terms of one unit, it is a simple matter to find its value in terms of a second unit. Thus to find the number of square feet in 3210 sq. in., since 1 square foot = 144 square inches, write

$$\frac{1}{144} = \frac{\text{no. of sq. ft.}}{\text{no. of sq. in.}} = \frac{x}{3210};\ \text{hence}$$

to 144 on *D*, set index of *C*,
opposite 3210 on *D*, read x = 22.3 on *C*;
that is, there are 22.3 sq. ft. in 3210 sq. in.

Again consider the problem of finding the number of nautical miles in 28.5 ordinary miles. Since there are 5280 ft. in an ordinary mile and 6080 ft. in a nautical mile, write

$$\frac{5280}{6080} = \frac{\text{ft. in naut. mi.}}{\text{ft. in ord. mi.}} = \frac{x}{28.5},$$

make the corresponding setting and read x = 24.8 naut. mi.

EXERCISES

1. An inch is equivalent to 2.54 cm. Find the respective length in cm. of rods 66 in. long, 98 in. long, and 386 in. long. Note the proportion:

$$\frac{\text{in.}}{\text{cm.}} : \frac{1}{2.54} = \frac{66}{x} = \frac{98}{y} = \frac{386}{z}.$$

2. One yd. is equivalent to 0.9144 meters. Find the number of meters in a distance of (*a*) 300 yd. (*b*) 875 yd. (*c*) 2.78 yd.

$$\textit{Hint}: \frac{\text{yd.}}{\text{m}} : \frac{1}{0.914} = \frac{300}{x} = \frac{875}{y} = \frac{2.78}{z}.$$

3. If 7.5 gal. water weighs 62.4 lb., find the weight of (*a*) 86.5 gal. water, (*b*) 247 gal. water, (*c*) 3.78 gal. water.

4. 31 sq. in. is approximately 200 sq. cm. How many square centimeters in (*a*) 36.5 sq. in.? (*b*) 144 sq. in.? (*c*) 65.3 sq. in.?

5. If one horsepower is equivalent to 746 watts, how many watts are equivalent to (*a*) 34.5 horsepower? (*b*) 5280 horsepower? (*c*) 0.832 horsepower?

6. If one gallon is equivalent to 3790 cu. cm., find the number of gallons of water in a bottle which contains (*a*) 4250 cu. cm. (*b*) 9.68 cu. cm. (*c*) 570 cu. cm. of the liquid.

*A table of conversion factors appears in Appendix B.

7. The intensity of pressure due to a column of mercury 1 inch high (1 inch of mercury) is 0.49 lb. per sq. in. If atmospheric pressure is 14.2 lb. per sq. in. what is atmospheric pressure in inches of mercury? What is a pressure of 286 lb. per sq. in. in inches of mercury? What is a pressure of 128 inches of mercury in lb. per sq. in.?

8. If P_1 represents the pressure per square unit on a given quantity of a perfect gas and V_1 the corresponding volume, then for two states of the gas at the same temperature

$$\frac{P_1}{P_2} = \frac{V_2}{V_1}.$$

The volume of a gas at constant temperature and pressure 14.7 lb. per sq. in. is 125 cu. in. (*a*) Find the respective pressures at which the volumes of the gas are 300 cu. in., 250 cu. in., 75.0 cu. in. (*b*) Find the respective volumes of the gas under the pressures: 85 lb. per sq. in., 55 lb. per sq. in., 23 lb. per sq. in., 10 lb. per sq. in.

19. The *DI*, *CI*, and *CIF* (reciprocal) scales. The reciprocal of a number is obtained by dividing 1 by the number. Thus, $\frac{1}{2}$ is the reciprocal of 2, $\frac{2}{3}$ ($= 1 \div \frac{3}{2}$) is the reciprocal of $\frac{3}{2}$, and $\frac{1}{a}$ is the reciprocal of a.

The reciprocal scales *CI* and *CIF*, on the front face of the slide rule, and *DI*, on the reverse face, are marked and numbered like the *C*, *CF*, and *D* scales respectively but in the reverse (or inverted) order; that is, the numbers represented by the marks on these scales increase from right to left. The red numbers associated with the reciprocal scales enable the operator to recognize these scales.

Rule. *When the hairline is set to a number on the C scale, the reciprocal (or inverse) of the number is at the hairline on the CI scale; conversely, when the hairline is set to a number on the CI scale, its reciprocal is at the hairline on the C scale.*

The same relation exists between the *D* and *DI* scales and between the *CF* and *CIF* scales.

To fix this relation in mind push the hairline in succession to the

D	1	0.5 (=1/2)	0.25 (=1/4)	0.2 (=1/5)	0.125 (=1/8)	0.1111 (=1/9)
DI	1	2	4	5	8	9

numbers on DI in the second row of the diagram and read on D the respective reciprocals written in the first row. Also opposite the numbers 1, 2, 4, 5, 8, and 9 on CI read their respective reciprocals on C. Again find the same opposites on CIF and CF.

By using the facts just mentioned, we can multiply a number or divide it by the reciprocal of another number. Thus to find $\frac{28}{7}$, we may think of it as $28 \times \frac{1}{7}$ and

to 28 on D set index of C,
opposite 7 on CI, read 4 on D.

Again to find 12×3, we may think of it as $12 \div \frac{1}{3}$ and

push hairline to 12 on D,
draw 3 of CI under the hairline,
opposite index of C, read **36** on D.

When the CI scale is used in multiplication and division, the position of the decimal point is determined in the usual way.

The DF and CIF scales may be used to perform multiplications and divisions in the same manner as the D and CI scales; thus to multiply 40.3 by 1/9.04,

to 403 on DF set index of CF,
opposite 904 on CIF, read **4.46** on DF.

Again to multiply 40.3 by 1/0.207,

to 403 on D set left index of C,
opposite 207 on CIF, read **194.7** on DF.

It should be noted that when the hairline is set to any number on a scale on one face of the slide rule, the rule may be turned over, without changing the position of the indicator, to read the opposite number on a scale on the other face of the slide rule.

EXERCISES

1. Use the DI scale to find the reciprocals of 16, 260, 0.72, 0.065, 17.4, 18.5, 67.1.

2. Find 18.2×21.7 in the usual way and then read $1/(18.2 \times 21.7)$ on DI opposite the first answer on D. Similarly find the values of $1/(2.87 \times 623)$, and $1/(0.324 \times 0.497)$.

3. Using the D scale and the CI scale, multiply 18 by 1/9 and divide 18 by 1/9.

4. Using the D scale and the CI scale, multiply 28.5 by 1/0.385 and divide 28.5 by 1/0.385. Also find 28.5/0.385 and 28.5×0.385 by using the C scale and the D scale.

5. Using the D scale and the CI scale, multiply 41.3 by 1/0.207 and divide 41.3 by 1/0.207.

6. Perform the operations of Exercises 2, 3, and 4 by using the CIF scale and the DF scale.

7. Set the hairline to 8.62 on DF, and read at the hairline $8.62/\pi$ on D and $\pi/8.62$ on DI. Also find the values of $1.23/\pi$, $\pi/1.23$, $39.4/\pi$, and $\pi/39.4$.

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20. Proportions involving the reciprocal scales. The reciprocal scales may be used in connection with proportions containing reciprocals. Since any number $a = 1 \div \frac{1}{a}$ and since $\frac{1}{a} = \frac{1}{a} \div 1$, we have

Rule (*C*). *The value of any ratio is not changed if any factor of its numerator be replaced by 1 and its reciprocal be written in the denominator, or if any factor of its denominator be replaced by 1 and its reciprocal be written in the numerator.* Thus $\frac{a}{b} = a\left(\frac{1}{b}\right) = \frac{1}{b\,(1/a)}$. Hence if $\frac{x}{a} = bc$, we may write $\frac{x}{a} = \frac{b}{(1/c)} = \frac{c}{(1/b)}$; if $ax = bc$, we may write $\frac{x}{(1/a)} = \frac{b}{(1/c)} = \frac{c}{(1/b)}$. A few examples will indicate the method of applying these ideas in computations.

To find the value of y which satisfies $\frac{y}{4.27} = 0.785 \times 3.76$, apply Rule ($C$) to get $\frac{D}{C}$: $\frac{y}{4.27} = \frac{0.785}{(1/3.76)}$.

Since, when 3.76 of CI is under the hairline, 1/3.76 of C is also under the hairline,

push hairline to 785 on D,
draw 376 of CI under the hairline,
opposite 427 on CF, read y = **12.60** on DF.

The position of the decimal point was determined by observing that y was nearly $4 \times 1 \times 4 = 16$.

To find the value of y which satisfies $1/(7.89y) = 0.381/0.0645$, use Rule (C) to obtain $\frac{D}{C}$: $\frac{(1/y)}{7.89} = \frac{0.381}{0.0645}$, and

push hairline to 381 on D,
draw 645 of C under the hairline,
opposite 7.89 on C, read $y = $ **0.0215** on DI.

The position of the decimal point was obtained by observing that 0.381 is about 6×0.06 and therefore that $1/y$ is about 6×8, or 48, so $y = 0.02$ approximately.

To find the values of x and y which satisfy $57.6x = 0.846y = 7$, use Rule (C) to obtain

$$\frac{D}{CI}: \quad \frac{x}{(1/57.6)} = \frac{y}{(1/0.846)} = \frac{7}{1}, \quad \text{and}$$

to 7 on D set index of CI,
opposite 576 on CI, read $x = $ **0.1215** on D,
opposite 846 on CIF, read $y = $ **8.27** on DF.

The folded scales may also be used. Thus to solve the same equation,

to 7 on DF set index of CIF,
opposite 576 on CIF, read $x = $ **0.1215** on DF,
opposite 846 on CIF, read $y = $ **8.27** on DF.

EXERCISES

In each of the following equations find the values of the unknown numbers:

1. $3.3x = 4.4y = \frac{75.2}{1.342}$.
2. $76.1x = 3.44y = \frac{111}{22.8}$.
3. $1.83x = \frac{y}{24.5} = (162)(1.75)$.
4. $\frac{0.342}{x} = \frac{y}{4.65} = (189)(0.734)$.
5. $5.83x = 6.44y = \frac{12.6}{z} = 0.2804$.
6. $3.42x = \frac{1.83}{y} = \frac{17.6}{z} = (2.78)(13.62)$.

21. Combined operations involving the reciprocal scales. The reciprocal scales may be used with scales C, D, CF and DF in combined operations involving a series of multiplications and divisions. In this connection the application of Rule (C) §20 will be helpful.

Example 1. Find the value of $1.843 \times 92 \times 2.45 \times 0.584 \times 365$, and of 1 divided by this product.

Solution. By using Rule (C) of §20, write the given expression in the form

$$\frac{1.843 \times 2.45 \times 365}{(1/92)(1/0.584)}$$

and reason as follows: (a) divide 1.843 by (1/92), (b) multiply the result by 2.45, (c) divide this second result by (1/0.584), (d) multiply the third result by 365. This argument suggests that we

push hairline to 1843 on D,
draw 92 of CI under the hairline,
push hairline to 245 on C,
draw 584 of CI under the hairline,
push hairline to 365 on C,
at the hairline read 886 on D, and 1129 on DI.

To approximate the first answer we write $\frac{2 \times 2 \times 400}{0.01 \times 2} = 80{,}000$.

Hence the answers are **88,600** and **0.00001129.**

Example 2. Find the value of

$$1/(352 \times 621 \times 0.0154 \times 0.00392).$$

Solution. This computation could be made by computing the denominator by a series of multiplications and then reading the reciprocal of the denominator on the DI scale. However the use of reciprocal scales in a combined operation is effective. Hence write the given expression in the form

$$\frac{(1/352)\ (1/0.0154)}{621 \times 0.00392};$$

push hairline to 352 on DI,
draw 621 of C under the hairline,
push hairline to 154 on CI,
draw 392 of C under the hairline,
opposite index of C, read 758 on D.

To approximate the answer write $1/(300 \times 600 \times .02 \times .004) = 1/14$(nearly) $= .07$(nearly). Therefore the answer is **0.0758.**

The following rule summarizes the process:

Rule. *To compute a number defined by a series of multiplications and divisions:*

(*a*) *arrange the expression in fractional form with one more factor in the numerator than in the denominator* (1 *may be used if necessary*),

(*b*) *push the hairline to the first number in the numerator on the D or DI scale,*

(*c*) *using the C or CI scale, take the other numbers alternately, drawing each number of the denominator under the hairline, and pushing the hairline to each number of the numerator,*

(*d*) *read the answer on the D scale,*

(*e*) *to get an approximation, compute the value of the expression obtained by replacing each number of the given expression by a convenient approximate number involving one, or at most two, significant figures.*

When necessary, interchange the indexes to make a setting possible. Also, the folded scales may be used to avoid shifting the slide. At any time the hairline may be pushed to a number on *C* or on *CF*; it is a good plan in combined-operation problems always to follow the operation of pushing the hairline to a mark on *C* or *CF* by drawing a mark of the same scale under the hairline.*

When a problem involving combined operations contains π as a factor, the statements dealing with π in §11 can be used in the solution.

It is interesting to observe that, when an answer is read on *D*, its reciprocal can be read at once opposite this answer on *DI*.

*In the combined-operation computation considered above, the scale of operation may be changed at will from the *C* scale to the *CF* scale or vice versa. In general, however, if the answer is read on the *D* scale, the number of times the hairline has been pushed to a mark on *CF* must be the same as the number of times a mark on *CF* has been drawn under the hairline. If the answer is read on *DF*, the process of pushing the hairline to a number on *CF* must have been used exactly one more time than the process of drawing a mark of *CF* under the hairline.

EXERCISES

1. $\dfrac{7 \times 8}{5}$.
2. $\dfrac{11 \times 12 \times 1}{7 \times 8}$.
3. $\dfrac{9 \times 7 \times 1}{8 \times (1/5)}$.
4. $\dfrac{1375 \times 0.0642}{76{,}400}$.
5. $\dfrac{45.2 \times 11.24}{336}$.
6. $\dfrac{218}{4.23 \times 50.8}$.

7. $\dfrac{235}{3.86 \times 3.54}$.
8. $2.84 \times 6.52 \times 5.19$.
9. $9.21 \times 0.1795 \times 0.0672$.
10. $37.7 \times 4.82 \times 830$.
11. $\dfrac{65.7 \times 0.835}{3.58}$.
12. $\dfrac{362}{3.86 \times 9.61}$.
13. $\dfrac{24.1}{261 \times 32.1}$.
14. $\dfrac{75.5 \times 63.4 \times 95}{3.14}$.
15. $\dfrac{3.97}{51.2 \times 0.925 \times 3.14}$.
16. $\dfrac{47.3 \times 3.14}{32.5 \times 16.4}$.
17. $\dfrac{3.82 \times 6.95 \times 7.85 \times 436}{79.8 \times 0.0317 \times 870}$.
18. $187 \times 0.00236 \times 0.0768 \times 1047 \times 3.14$.
19. $\dfrac{0.917 \times 8.65 \times 1076 \times 3152}{7840}$.
20. $\dfrac{45.2 \times 11.24\pi}{336}$.
21. $\dfrac{45.2 \times 11.24}{336\pi}$.

In evaluating the exercises numbered 22–25, compute the denominations by straight multiplication and read the reciprocals of the denominators on the *DI* scale.

22. $\dfrac{1}{421 \times 632}$.
23. $\dfrac{1}{827 \times 6.28 \times 273}$.
24. $\dfrac{1}{0.153 \times 0.646 \times 5.72 \times 0.628}$.
25. $\dfrac{1}{3.14 \times 2.72 \times 1.414 \times 1.572}$.
26. Solve problems 22–25 by using the method of Example 2.

22. Reciprocal scales in electrical engineering calculations. Many formulas in electrical engineering take the form $\dfrac{1}{lmn}$; that is, a fraction with 1 as the numerator and the product of three numbers in the denominator. For example, radio and television engineers frequently need to calculate the reactance of a capacitance to the flow of alternating current. The formula is:

$$\text{Reactance} = \frac{1}{6.28fC}$$

where f = frequency and C = capacitance.

Expressions of the form $\dfrac{1}{lmn}$ can be evaluated by using scales *C*, *CI* and *D* to obtain the product $l \times m \times n$, but instead of reading this product under the hairline on *D*, read its reciprocal under the hairline on DI. The procedure is illustrated in the following setting:

push hairline to l on D,
draw m of CI under the hairline,
push hairline to n on C,
under hairline on DI read $\frac{1}{lmn}$.

Suppose, as is often the case, that one wishes to find the reactance at a given frequency of several capacitances connected in series. The formula is then:

$$\text{Reactance} = \frac{1}{6.28fC_1} + \frac{1}{6.28fC_2} + \dots \frac{1}{6.28fC_n}.$$

This can be written in the form:

$$\text{Reactance} = \frac{1}{6.28f}\left(\frac{1}{C_1} + \frac{1}{C_2} + \dots \frac{1}{C_n}\right).$$

This expression can be evaluated quite simply on the slide rule by first finding the reciprocal of C_1, the reciprocal of C_2, . . . the reciprocal of C_n, and adding these reciprocals to obtain the sum S. Then divide S by $6.28f$.

To find the value of the reciprocal of C_1,
push hairline to C_1 on D,
under hairline read on DI the reciprocal of C_1.

A similar setting is used to find each of the remaining reciprocals.

To find the value of $\frac{S}{6.28f}$

push hairline to S on D,
draw 628 of C under the hairline,
push hairline to f on CI,
under hairline read **answer** on D.

Another useful example is computing the total reactance of a circuit containing inductance L and capacitance C in series. The equation is:

$$\text{Reactance} = 6.28fL - \frac{1}{6.28fC}.$$

The procedure is:

push hairline to 6.28 on D,
draw f of CI under the hairline,
push hairline to L on C,
under hairline read, on D, $6.28fL$;
push hairline to C on Scale C,
under hairline read, on DI, $\frac{1}{6.28fC}$.
Subtract the second result from the first.

EXERCISES

Evaluate the expression

$$l\,m\,n - \frac{1}{l\,m\,p}$$

for the sets of values numbered 1-6:

1. $l = 5.41$, $m = 3.14$, $n = 0.226$, $p = 0.0635$.
2. $l = 0.759$, $m = 60.1$, $n = 0.154$, $p = 0.00632$.
3. $l = 6.28$, $m = 54.2$, $n = 0.0246$, $p = 0.00542$.
4. $l = 6.28$, $m = 63.2$, $n = 0.562$, $p = 0.0000653$.
5. $l = 6.28$, $m = 60.0$, $n = 0.247$, $p = 15.00 \times 10^{-5}$.*
6. $l = 6.28$, $m = 63.5$, $n = 0.152$, $p = 5.16 \times 10^{-5}$.

Evaluate the expression

$$\frac{1}{lm}\left(\frac{1}{n_1} + \frac{1}{n_2} + \dots \frac{1}{n_k}\right)$$

for the sets of values numbered 7-12:

7. $l = 2.50$, $m = 40.1$, $n_1 = 0.641$, $n_2\,(=n_k) = 1.03$.
8. $l = 3.65$, $m = 30.6$, $n_1 = 0.00347$, $n_2 = 0.00297$, $n_3\,(=n_k) = 0.00472$.
9. $l = 6.28$, $m = 60.0$, $n_1 = 0.000346$, $n_2 = 0.000463$, $n_3\,(=n_k) = 0.000645$.
10. $l = 6.28$, $m = 61.3$, $n_1 = 3.42 \times 10^{-6}$, $n_2 = 2.71 \times 10^{-6}$, $n_3\,(=n_k) = 5.62 \times 10^{-6}$.
11. $l = 6.28$, $m = 62.4$, $n_1 = 3.01 \times 10^{-6}$, $n_2 = 5.62 \times 10^{-6}$, $n_3\,(=n_k) = 5.81 \times 10^{-6}$.
12. $l = 6.28$, $m = 40.3$, $n_1 = 5.21 \times 10^{-6}$, $n_2 = 8.21 \times 10^{-6}$, $n_3\,(=n_k) = 7.51 \times 10^{-6}$.

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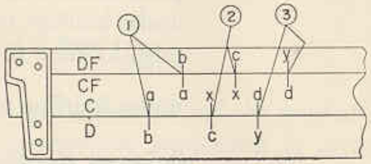
*For an explanation of the powers-of-ten notation, see Article 67.

23. Visual summary.

Proportion: $\frac{a}{b} = \frac{x}{c} = \frac{d}{y}$.

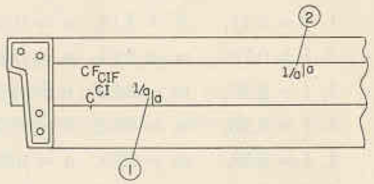
1. To b on D (DF) set a on C (CF),
2. at c on D (DF) read x on C (CF),
3. at d on C (CF) read y on D (DF).



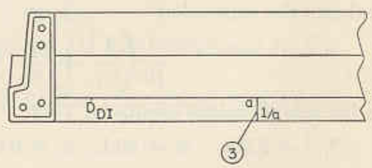
Be sure to keep all denominators on body and all numerators on slide.

Reciprocal: $x = \frac{1}{a}$.

1. At a on C read $\frac{1}{a}$ on CI,
2. at a on CF read $\frac{1}{a}$ on CIF,



3. at a on D read $\frac{1}{a}$ on DI.



CHAPTER III

SQUARES AND SQUARE ROOTS; CUBES AND CUBE ROOTS

24. Introduction. This chapter explains how to use the slide rule for solving problems involving squares, square roots, cubes and cube roots. In this connection, the slide rule is used in two basic ways: (1) as a table to determine the values of such functions, and (2) as a computer to perform calculations involving them. The scales here considered are the *square scales Sq1* and *Sq2*, the *square root scales A* and *B*, and the *cube root scale K*.

The DECI-LON slide rule may be used with greater facility and accuracy in certain problems by virtue of having the *Sq1* and *Sq2* scales in addition to the time-tested powerful *A* and *B* scales.

Scales *A* and *B* are used for multiplications, divisions, and combined operations involving primarily square roots. In addition, however, many combined operations involving squares may be performed by using scales *A* and *B*.

Scales *Sq1* and *Sq2* are double unit length scales. They are especially adapted for obtaining the values of squares and square roots, and are also useful in performing directly certain frequently used calculations involving squares, such as finding the areas of circles.

A complete explanation of the principles of the square, square root, and cube root scales is given in Chapter VII, §§ 87 and 88.

25. *Sq1* and *Sq2* scales; squares. The square of a number is the result of multiplying the number by itself. Thus $2^2 = 2 \times 2 = 4$.

Scales *Sq1* and *Sq2* together will be referred to as the **square scales** (either one singly as a **square scale**).

The square scales (*Sq1*, *Sq2*) are so constructed that:

Rule. *When the hairline is set to a number on scale Sq1 or Sq2, the square of the number is found under the hairline on scale D.*

To gain familiarity with the relations between these scales consider the following examples:

To find 2^2, push hairline to 2 on *Sq1*,
under hairline read **4** on *D*.

To find 4^2, push hairline to 4 on *Sq2*,
under hairline read **16** on *D*.

To find 278^2, push hairline to 278 on *Sq1*,
under hairline read **773** on *D*.

To determine the position of the decimal point, round off the given number to 300 and note that $(300)^2 = 90{,}000$. Hence the answer is **77,300**. In a great many cases the method of approximating the answer as demonstrated above can be used to advantage.

The following rules for determining the position of the decimal point in squaring a number will be helpful. Rule 1 refers to numbers greater than 1, Rule 2 to numbers less than 1.

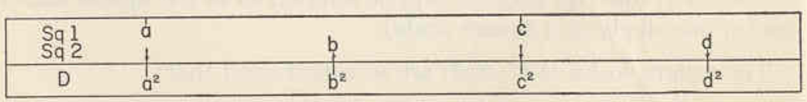
Rule 1. Squaring a number greater than 1. *Let n denote the number of digits to the left of the decimal point. If the number to be squared occurs on Sq1, the square of the number will contain (2n-1) digits to the left of the decimal point. If the number occurs on Sq2, the square will contain 2n digits to the left of the decimal point.*

In Rule 2, for numbers less than 1, the zeros between the decimal point and the first non-zero digit are called **significant zeros.**

Rule 2. Squaring a number less than 1. *Let m denote the number of significant zeros. If the number to be squared occurs on Sq1, the square of the number will contain (2m + 1) significant zeros. If the number occurs on Sq2, the square will contain 2m significant zeros.*

These rules are visually summarized in Fig. 13.

Numbers greater than 1 with n digits left of decimal point	Numbers less than 1 with m significant zeros



$(2n-1)$ digits left of decimal point	$2n$ digits left of decimal point	$(2m+1)$ significant zeros	$2m$ significant zeros

FIG. 13.

Example 1. Find $(260)^2$.

Solution. Push hairline to 26 on *Sq1*,
under hairline read **67,600** on *D*.

Note that the given number occurs on *Sq1* and contains three digits to the left of the decimal point. Hence $n = 3$ and $(2n\text{-}1) = 5$. Therefore the square must contain five digits to the left of the decimal point.

Example 2. Find $(0.0610)^2$.

Solution. Push hairline to 61 on *Sq2*,
under hairline read **0.00372** on *D*.

Note that the given number is less than 1, contains one significant zero, and occurs on *Sq2*. Hence $m = 1$, $2m = 2$ and the square must contain two significant zeros.

26. Area of a circle.* The area of a circle may be conveniently found when its radius is known by using the square scales in conjunction with the *DF* scale. The formula for the area A of a circle in terms of its radius r is $A = \pi r^2$. Recalling that each number on *DF* is π times its opposite on *D*, to find the area of a circle it is only necessary to

push hairline to radius on an *Sq* scale,
under hairline on *DF* read the area of the circle.

Example. Find the area of a circle of 8.7 ft. radius.

Solution. In accord with the above setting,
push hairline to 87 on *Sq2*,
under hairline read **238** on *DF*.

Therefore the area is **238** sq. ft. The decimal point was placed in accord with Rule 1 above.

Engineers generally use the formula for the area A of a circle in terms of its diameter d, namely

$$A = \frac{\pi d^2}{4}.$$

The following example covers this case:

Example. Find the area of a circle having 3.61 in. diameter.

* See Appendix A for an explanation of the use of the $\frac{\pi}{4}$ locating mark as a quick method of finding areas of circles when diameter is given.

Solution. The formula for the area A of a circle, adapted to the given data, yields $A = \frac{\pi(3.61)^2}{4}$. Hence

push hairline to 361 on *Sq2*,
draw 4 of C under hairline,
interchange the indexes,
opposite π on CF read **10.24** on DF.

EXERCISES

1. Use the slide rule to find the square of each of the following numbers: 25, 32, 61, 75, 89, 733, 452, 2.08, 1.753, 0.334, 0.00356, 0.953, 5270, 4.73×10^6.*
2. Find the area of a circle having radius (*a*) 3.46 ft. (*b*) 0.0436 ft. (*c*) 17.53 ft. (*d*) 8650 ft.
3. Find the area of a circle having diameter (*a*) 2.75 ft. (*b*) 66.8 ft. (*c*) 0.753 ft. (*d*) 1.876 ft.

27. Evaluation of simple expressions involving squares. When the hairline is set to a number on a square scale, its square is automatically read under the hairline on the D scale. Consequently many expressions involving squares can be evaluated conveniently. Thus to find $x = \frac{(24.6)^2 \times (0.785)}{4.39}$

push hairline to 246 on *Sq1*,
draw 439 of C under the hairline,
push hairline to 785 on CF,
under hairline read **108.0** on DF.

EXERCISES

1. $\frac{(2.38)^2\ 19.7}{18.14}$.
2. $\frac{5.66\ (7.48)^2}{79}$.
3. $\frac{6.76}{2.17\ (2.7)^2}$. Hint: first find $\frac{2.17\ (2.7)^2}{6.76}$ and then find its reciprocal.
4. $\frac{2.56 \times 4.86}{(1.365)^2}$.
5. $\frac{(2.60)^2}{2.17 \times 7.28}$.

* In dealing with combinations of very large numbers or very small numbers it is advisable to use the powers-of-ten notations in placing the decimal point. Article 67 indicates the method to be used.

28. Scales *A* and *B*; square roots.* The square root of a given number is a number whose square is the given number. Thus the square root of 4 is 2 and the square root of 9 is 3, or, using the symbol for square root, $\sqrt{4} = 2$, and $\sqrt{9} = 3$.

Scales A and B will be referred to as the **square root scales.**

Scale A consists of two parts which differ only in slight details. We shall refer to the left hand part as *A left* and to the right hand part as *A right*. Similar reference will be made to the B scale. Scale A is on the body. Scale B is on the slide. In all other respects the two scales are identical.

The A scale is so designed that when the hairline is set to a number on the A scale the square root of the number is on scale D under the hairline. Hence:

Rule. *To find the square root of a number between 1 and 10 push the hairline to the number on A left and read its square root under the hairline on scale D. To find the square root of a number between 10 and 100 push the hairline to the number on A right and read its square root under the hairline on scale D. In either case place the decimal point in the square root after the first digit.*

This statement also applies if A and D are replaced by B and C respectively; i.e., a square root may be determined by using the A and D scales on the body, or the B and C scales on the slide.

As an illustration of the above statement consider the following:

To find the square root of 9.00,
push hairline to 9 on *A left*,
under hairline read **3.00** on D.

To find the square root of 16.00,
push hairline to 16 on *A right*,
under hairline read **4.00** on D.

* See Appendix A for the use of π and $\frac{\pi}{4}$ marks on A and B scales.

When a number is outside the range from 1 to 100, its square root can still be found on the slide rule by using the following fact: moving the decimal point **two** places in a **number** results in moving the decimal point **one** place in the **square root** of the number. Hence:

Rule. *To obtain the square root of any number outside the range of 1 to 100, move the decimal point an* **even number** *of places to obtain a number between 1 and 100, find the square root of this latter number, then move the decimal point in this square root* **one half** *as many places as it was moved in the original number but in the* **opposite** *direction.*

Example 1. Find $\sqrt{432}$.

Solution. Move the decimal point two places to the **left** to obtain $\sqrt{4.32}$, and

push hairline to 432 on *A left*,
under hairline read 208 on *D*.

Therefore $\sqrt{4.32}$ is 2.08. Finally, since the decimal point was moved **two** places to the **left** in the original number, move the decimal point in this last result **one** place to the **right** to obtain the answer **20.8**.

Example 2. Find $\sqrt{0.432}$.

Solution. Move the decimal point **two** places to the **right** to obtain $\sqrt{43.2}$, and

push hairline to 432 on *A right*,
under hairline read 658 on *D*.

Therefore $\sqrt{43.2} = 6.58$; finally move the decimal point in this last result **one** place to the **left** to obtain the answer **0.658**.

EXERCISES

1. Find the square root of each of the following numbers: 8, 12, 17, 89, 8.90, 890, 0.89, 7280, 0.0635, 0.0000635, 63,500, 100,000.

2. Find the length of the side of a square whose area is (*a*) 53,500 ft.^{2}; (*b*) 0.0776 ft^2; (*c*) 3.27×10^7ft.^{2}.

3. Find the diameter of a circle having area (*a*) 256 ft.^{2}; (*b*) 0.773 ft.^{2}; (*c*) 1950 ft.^{2}

29. Combined operations involving square roots. The principles explained in §§12 and 21 may be applied to evaluate a fraction containing indicated square roots as well as numbers and reciprocals of numbers. If the reader will recall that when the hairline is set to a number on the *CI* scale it is automatically set to the reciprocal of the number on the *C* scale and when set to a number on the *B* scale it is automatically set to the square root of the number on the *C* scale, he will easily understand that the method used in this article is essentially the same as that used in §21. The principle of determining whether *B left* or *B right* should be used is the same whether we are merely extracting the square root of a number or whether the square root is involved with other numbers.

Example 1. Evaluate $\dfrac{915 \times \sqrt{36.5}}{804}$.

Solution. Remembering that the hairline is automatically set to $\sqrt{36.5}$ on the *C* scale when it is set to 36.5 on *B right*, use the rule of §12 and

push hairline to 915 on *D*,
draw 804 of the *C* scale under the hairline,
push hairline to 365 on *B right*,
under hairline read **6.88** on *D*.

Example 2. Evaluate $\dfrac{\sqrt{832} \times \sqrt{365} \times 1863}{(1/736) \times 89{,}400}$.

Solution. Before making the setting indicated in this solution, read the italicized rule in §21.

Push hairline to 832 on *A left*,
draw 736 of *CI* under the hairline,
push hairline to 365 on *B left*,
draw 894 of *C* under the hairline,
push hairline to 1863 on *CF*,
under hairline read **8450** on *DF*.

To get an approximate value write $\dfrac{(30)\ (18)\ (2000)\ (700)}{90{,}000} = 8400.$

Example 3. Evaluate $\dfrac{0.286 \times 652 \times \sqrt{2350} \times \sqrt{5.53}}{785\ \sqrt{1288}}$.

Solution. Write the expression in the form

$$\frac{0.286 \times \sqrt{2350} \times \sqrt{5.53} \times 1}{(1/652 \times 785 \times \sqrt{1288}}$$

Push hairline to 286 on D,
draw 652 of CI under the hairline,
push hairline to 235 on B *right*,
draw 785 of C under the hairline,
push hairline to 553 on B *left*,
draw 1288 of B *righi* under hairline,
opposite the index of C, read 0.755 on D.

As an approximate value use $\frac{.3\ (700)\ (50)\ (2)}{800(30)} = 0.9.$

EXERCISES

1. $42.2\sqrt{0.328}$.
2. $1.83\sqrt{0.0517}$.
3. $\sqrt{3.28} \div 0.212$.
4. $\sqrt{51.7} \div 103$.
5. $0.763 \div \sqrt{0.0296}$.
6. $\frac{\sqrt{277}}{5.34 \times \sqrt{7.02}}$.
7. $\frac{645}{5.34\sqrt{13.6}}$.
8. $14.3 \times 47.5\sqrt{0.344}$.
9. $20.6 \times \sqrt{7.89} \times \sqrt{0.571}$.
10. $\frac{7.92\sqrt{7.89}}{\sqrt{0.571}}$.
11. $\frac{7.87 \times \sqrt{377}}{2.38}$.
12. $\frac{86 \times \sqrt{734} \times \pi}{775 \times 0.685}$.
13. $\frac{4.25 \times \sqrt{63.5} \times \sqrt{7.75}}{0.275 \times \pi}$.
14. $\frac{189.7 \times \sqrt{0.00296} \times \sqrt{347} \times 0.274}{\sqrt{2.85} \times 165 \times \pi}$.
15. $\sqrt{285} \times 667 \times \sqrt{6.65} \times 78.4 \times \sqrt{0.00449}$.
16. $\frac{239 \times \sqrt{0.677} \times 374 \times 9.45 \times \pi}{84.3 \times \sqrt{9350} \times \sqrt{28400}}$.

30. *K* scale; cube roots. The cube of a number is the result of using the number three times as a factor. Thus the cube of 3 (written 3^3) is $3 \times 3 \times 3 = 27$.

The cube root of a given number is a number whose cube is the given number. Thus the cube root of 27 (written $\sqrt[3]{27}$) is 3, since $3^3 = 27$. The $\sqrt[3]{64}$ is 4 since $4^3 = 64$.

The K scale is on the body. It is divided into three equal and identical sections. We shall refer to the left hand section, the middle section, and the right hand section as K *left*, K *middle* and K *right* respectively.

The K scale is so constructed that when the hairline is set to a number on the K scale the cube root of the number is on the D scale at the hairline.

Rule. *To find the cube root of a number between 1 and 10, set the hairline to the number on K left and read its cube root at the hairline on D. To find the cube root of a number between 10 and 100, set the hairline to the number on K middle and read its cube root at the hairline on D. The cube root of a number between 100 and 1000 is found on the D scale opposite the number on K right.*

In each of these three cases the decimal point is placed after the first digit.

As an illustration of the above statement consider the following:

To find the cube root of 27,
push hairline to 27 on K *middle*,
under hairline read 3 on D.

To find the cube root of 343,
push hairline to 343 on K *right*,
under hairline read 7 on D.

When a number is outside the range from 1 to 1000, its cube root can be found on the slide rule by making use of the following fact: moving the decimal point **three** places in a **number** results in moving the decimal point **one** place in the **cube root** of the number. Hence:

Rule. *To obtain the cube root of a number outside the range from 1 to 1000, move the decimal point* **three places** *at a time until a number between 1 and 1000 is obtained. Find the cube root of this latter number, then move the decimal point in the cube root* **one third** *as many places as it was moved in the original number but in the* **opposite** *direction.*

Example 1. Find $\sqrt[3]{23{,}400{,}000}$.

Solution. Move the decimal point 6 places to the **left,** thus obtaining 23.4. Since this is between 1 and 1000,

push hairline to 23.4 on *K middle*,
under hairline read on *D*, $2.86 = \sqrt[3]{23.4}$,
move the decimal point $\frac{1}{3}$ (6) = 2 places to
the *right* to obtain the answer, **286**.

The decimal point could have been placed by observing that $\sqrt[3]{27{,}000{,}000} = 300$.

Example 2. Find $\sqrt[3]{0.000585}$.

Solution. Move the decimal point 6 places to the **right** to obtain $\sqrt[3]{585}$, a number between 1 and 1000, and

push hairline to 585 on *K right*,
under hairline read on *D*, $8.36 = \sqrt[3]{585}$,
move the decimal point $\frac{1}{3}$ (6) = 2 places to
left to obtain the answer, **0.0836**.

EXERCISES

Find the cube root of each of the following numbers: 8.72, 30, 729, 850, 7630, 0.00763, 0.0763, 0.763, 89,600, 0.625, 75×10^7, 10, 100, 100,000.

31. Cubes, using *K* scale.

By interchanging the roles of the *K* and *D* scales in the operations performed in the preceding article for finding cube roots, we may find the cubes of numbers using scales *K* and *D*. In this connection the following rule may be found helpful.

Rule. *To find the cube of a number, set the hairline to the number on the D scale, and read its cube on the K scale at the hairline.*

To convince himself of this, the reader should set the hairline to 2 on *D* and read $2^3 = 8$ at the hairline on *K*; set the hairline to 3 on *D* and read $3^3 = 27$ at the hairline on *K*, etc. To find 21.7^3, set the hairline to 217 on *D* and read 102 on *K*. Since $20^3 = 8000$, the answer is near 8000. Hence we write **10,200** as the answer. To obtain this answer otherwise, write

$$21.7^3 = \frac{21.7 \times 21.7}{(1/21.7)} = \mathbf{10{,}220}$$

and use the general method of combined operations. This latter method is more accurate as it is carried out on the full length scales.

EXERCISES

1. Cube each of the following numbers by using the *K* scale and also by using the method of combined operations: 2.1, 3.2, 62, 75, 89, 733, 0.452, 3.08, 1.753, 0.0334, 0.943, 5270, 3.85×10^6.

2. How many gallons will a cubical tank hold that measures 26 inches in depth? (1 gal. = 231 cu. in.)

32. Additional use of the square root scales *A* and *B*.

Squaring a number is the inverse operation to extracting its square root. It is not surprising therefore to find that squares can be obtained using the square root scales *A* and *B*. In this connection the following rule will be found useful.

Rule. *To find the square of a number using scales A and D, set the hairline to the number on scale D, and under the hairline read on scale A the square of the number. Similarly, to find the square of a number using scales B and C, set the hairline to the number on scale C and under the hairline read on scale B the square of the number.*

To gain familiarity with this use of scales *A* and *B* make the following settings:

To find 3^2,

push hairline to 3 on *D*,
under hairline read **9** on *A*.

To find 4^2,

push hairline to 4 on *D*,
under hairline read **16** on *A*, or

push hairline to 4 on *C*,
under hairline read **16** on *B*.

Moreover, since scales *A* and *B* are identical scales, with scale *A* being on the body and scale *B* being on the slide, they can be used to multiply and divide numbers, just as scales *C* and *D* are used. Squares of numbers may be multiplied and divided using scales *A* and *B* in combined operations by noting that when an operation of multiplication and division of numbers is being performed on scales *D* and *C*, the same multiplication and division of their squares is automatically being performed on scales *A* and *B*.

Example. Evaluate $x = \dfrac{(3.7)^2\ (8.51)\ (9.61)}{(7.32)^2\ \ 32.8}$.

Solution. Divide 3.7 by 7.32 using scales *C* and *D*. By doing this, $(3.7)^2$ divided by $(7.32)^2$ was automatically obtained on scales *A* and *B*. From then on continue the multiplication and division using scales *A* and *B* as fundamental scales. The setting is as follows:

push hairline to 37 on *D*,
draw 732 of *C* to the hairline,
push hairline to 851 on *B left*,
draw 328 of *B right* to the hairline,
push hairline to 961 on *B left*,
under hairline read **0.637** on *A*.

The position of the decimal point was determined from the approximation:

$$\frac{4^2 \times 9 \times 9}{7^2 \times 30} = \frac{16 \times 9 \times 9}{49 \times 30} = \text{approx. } \frac{9}{10}.$$

EXERCISES

1. $\dfrac{(2.38)^2 \times 19.7}{18.14}$.
2. $\dfrac{5.66 \times (7.48)^2}{79}$.
3. $\dfrac{6.76}{2.17\ (2.7)^2}$. Hint: divide 6.76 by 2.17 using scales *A* and *B* and divide the result by $(2.7)^2$ using scale *CI*.
4. $\dfrac{2.56 \times 4.86}{(1.365)^2}$.
5. $\dfrac{20.6 \times (7.89)^2 \times (6.79)^2}{(467)^2 \times 281}$.

33. Further use of the square scales *Sq1* and *Sq2*. Square roots of numbers can be obtained by using the square scales *Sq1* and *Sq2* in an inverse operation.

For cases in which it is required to find the values of square roots independent of combined operations, scales *Sq1* and *Sq2* can be used to some advantage as a table of square roots, since the unit of measure of these scales is four times that of scales *A* and *B*.

The rule for finding square roots by means of the square scales *Sq1* and *Sq2* is here set forth:

Rule. *To find the square root of a number by means of the square scales Sq1 and Sq2:*

For a number between 1 and 10,
push hairline to the number on scale D,
under hairline read the square root on Sq1.

For a number between 10 and 100,
push hairline to the number on scale D,
under hairline read the square root on Sq2.

For example, to find $\sqrt{9}$,
push hairline to 9 on *D*,
under hairline read **3** on *Sq1*.

To find $\sqrt{25}$,
push hairline to 25 on *D*,
under hairline read **5** on *Sq2*.

To find the square root of any number outside the 1-to-100 range, move the decimal point as explained in §28.

The following is a useful rule for determining whether to use scale *Sq1* or *Sq2* when finding the square root of any number:

Rule. *To find the square root of a number greater than 1 use Sq1 when it contains an odd number of digits to the left of the decimal point; otherwise use Sq2.*

For a number less than 1 use Sq1 if the number of zeros immediately following the decimal point is odd; otherwise use Sq2.

Example. Find the square root of 24,300.

Solution. 24,300 has 5 digits to the left of the decimal point. Hence its square root must be read on scale *Sq1*. Accordingly,

push hairline to 243 on *D*,
under hairline read **156.0** on *Sq1*.

The position of the decimal point was determined by methods explained in §28.

EXERCISES

1. Find the square root of each of the following numbers: 64, 169, 91, 67.3, 4760, 476, 0.0721, 0.00764.

2. Find the radius of a circle having area (*a*) 42 sq. ft. (*b*) 167 sq. ft. (*c*) 83 sq. ft. (*d*) 192 sq. ft. (*e*) 456 sq. ft.

---

34. Combined operations. By setting the hairline to numbers on various scales we may set square roots, cube roots, and reciprocals of numbers on the D scale or on the C scale. Hence we can use the slide rule to evaluate expressions involving such quantities, and we can solve proportions involving them. The position of the decimal point is determined by an approximate calculation. When no confusion results, the student should always think of a combined operations problem as a series of multiplications and divisions, reserving the proportion principle for use in cases of doubt. The first two examples below are solved by the proportion principle, whereas Example 3 is considered as a series of multiplications and divisions.

Example 1. Find the value of $\dfrac{\sqrt[3]{385}}{2.36}$.

Solution. We may think of this as a division or as the proportion

$$\frac{x}{1} = \frac{\sqrt[3]{385}}{2.36}, \text{ and then}$$

push hairline to 385 on K *right*,
draw 236 of C under the hairline,
opposite index of C, read **3.08** on D.

Example 2. Find the value of $\dfrac{5.37\ \sqrt[3]{0.0835}}{\sqrt{52.5}}$.

Solution. Equating the given expression to x and applying Rule (B) §17 we write

$$\frac{x}{5.37} = \frac{\sqrt[3]{0.0835}}{\sqrt{52.5}}.$$

This proportion suggests the following setting:

push hairline to 835 on K *middle*,
draw 525 of B *right* under the hairline,
push hairline to 537 on C,
under hairline read **0.324** on D.

Example 3. Evaluate $\dfrac{(1.736)\ (6.45)\ \sqrt{8590}\ \sqrt[3]{581}}{\sqrt{27.8}}$

Solution. By using Rule (C) of §20, write the given expression in the form

$$\frac{\sqrt[3]{581}\ (6.45)\ \sqrt{8590}}{(1/1.736)\ \sqrt{27.8}}.$$

Push hairline to 581 on K *right*,
draw 1736 of CI under the hairline,
push hairline to 645 on C,
draw 278 of B *right* under the hairline,
push hairline to 8590 on B *right*,
under hairline read **1643** on D.

EXERCISES

1. $\sqrt[3]{73.2}\ (0.523)$.
2. $24.3\ \sqrt[3]{0.0661\pi}$.
3. $489 \div \sqrt[3]{732}$.
4. $27\pi \div \sqrt[3]{661{,}000}$.
5. $\sqrt[3]{531} \div \sqrt{28.4}$.
6. $\sqrt{9.80} \div \sqrt[3]{160{,}000}$
7. $(72.3)^2 \times 8.25$.
8. $\dfrac{\pi(0.213)^2}{0.0817}$.
9. $\dfrac{\sqrt[3]{19.2^2}}{(7.13)^2 \times 0.122}$.
10. $\dfrac{\pi\sqrt[3]{740}}{4.46 \times \sqrt{28.5}}$.
11. $3.83 \times 6\ 26 \times \sqrt[3]{54.2}$.
12. $0.437 \times \sqrt{564} \times \sqrt[3]{1.86}$.
13. $675 \times \sqrt{0.346} \times \sqrt[3]{0.00711}$.
14. $\dfrac{\sqrt[3]{32.1}\ (0.0585)\pi}{(1/3.63)}$.
15. $\dfrac{3.57 \times \sqrt{643} \times 4250}{0.0346\ \sqrt{0.00753}}$.
16. $\dfrac{\sqrt[3]{0.00335}\ \sqrt{273}}{787\ \sqrt{0.723}}$.
17. $\dfrac{0.0872 \times 36.8 \times \sqrt{2.85}}{0.343\pi}$
18. $76.2\ \sqrt[3]{56.1}\ \sqrt{877}\ (1/3.78)$.
19. $\dfrac{\sqrt{1.735}}{0.0276\ \sqrt{58{,}300} \times 7.63 \times 0.476}$.
20. $\dfrac{68.7\ \sqrt[3]{3160}\ \sqrt{0.0317} \times 89.3}{17.6 \times 277}$.
21. $\dfrac{\sqrt[3]{0.0645} \times 1834 \times \sqrt{21.6}}{89.6 \times 748 \times \sqrt{3460}}$.
22. $\sqrt{(27.5)^2 - (3.483)^2}$.

23. The maximum time E in hours that an airplane will remain aloft may be approximated by

$$E = \frac{750\ N\sqrt{w}\ \ L/D)}{CV_c} \left(\frac{1}{\sqrt{w_1}} - \frac{1}{\sqrt{w_0}}\right),$$

where the letters represent certain quantities for an airplane. Compute E if $w = 21500$, $V_C = 190$, $N = 0.85$, $C = 0.465$, $L/D = 20.0$, $w_1 = 19100$, $w_0 = 24000$.

Hint:

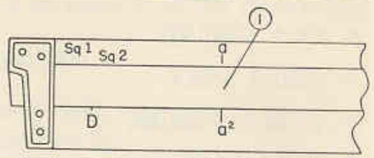
push hairline to 750 on D,
draw 190 of C under the hairline,
push hairline to 215 on B *left*,
draw 465 of C under the hairline,
push hairline to 0.85 on CF,
draw 20 of CIF under the hairline,
push hairline to 1 on C,
draw 191 of B *left* under the hairline,
opposite index of C, read on D 153.1 $= F_1$,
draw 24 of B *left* under the hairline,
opposite index of C, read on D 136.6 $= F_2$

and subtract F_2 from F_1 to get 16.5 hours.

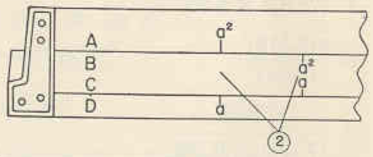
35. Visual summary.

Square of a number: $x = a^2$.

1. At a on $Sq1$ or $Sq2$ read a^2 on D; or

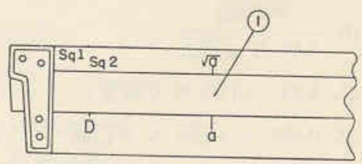


2. at a on D (C) read a^2 on A (B).

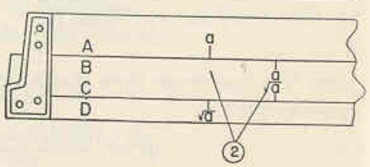


Square root of a number: $x = \sqrt{a}$.

1. At a on D read $\sqrt{a}$ on $Sq1$ or $Sq2$; or

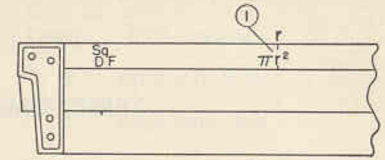


2. at a on A (B) read $\sqrt{a}$ on D (C).



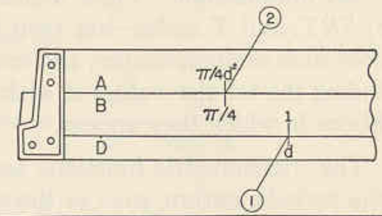
Area of a circle: $x = \pi r^2 = \frac{\pi}{4} d^2$.

1. At r on $Sq1$ ($Sq2$) read πr^2 on DF.



1. To d on D set slide index,

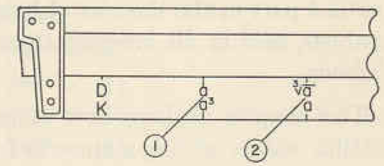
2. at $\frac{\pi}{4}$ on B read $\frac{\pi}{4} d^2$ on A.



Cube or cube root of a number: $x = a^3$ or $x = \sqrt[3]{a}$.

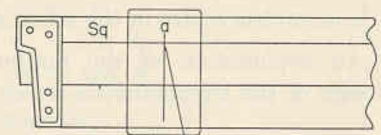
1. At a on D read a^3 on K,

2. at a on K read $\sqrt[3]{a}$ on D.



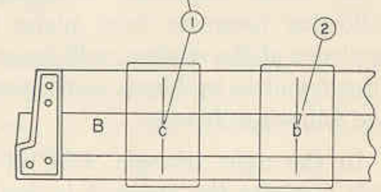
Squares and square roots in combined operation:

$$x = \frac{a^2 \sqrt{b}}{\sqrt{c}\sqrt{d}}.$$



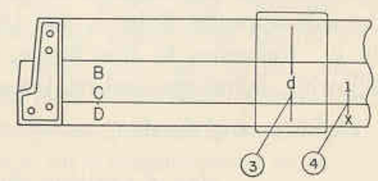
1. To a on $Sq1$ ($Sq2$) set c on B,

2. push hairline to b on B,



3. draw d of B under hairline,

4. at slide index read x on D.



CHAPTER IV

TRIGONOMETRIC SCALES

36. Introduction. Three scales on the DECI-LON slide rule—the *S*, *SRT*, and *T* scales—are used in dealing with trigonometric functions such as sines, cosines, and tangents. These scales can be used for finding the tabular values of such functions, or for performing calculations in which they appear.

The trigonometric functions are capable of describing phenomena of a periodic nature, such as the to-and-fro motion of a pendulum or the undulating motion of waves. Consequently, they play an important part in the theories of light and sound, in electricity, in wave analysis, and in all investigations of vibratory and oscillatory phenomena.

This chapter explains how to use the slide rule for determining the tabular values of trigonometric functions, and how to perform computations using these functions purely as numbers, as in equations of oscillatory phenomena. The next chapter tells how to use the trigonometric scales in the solution of triangles.

An explanation of the mathematical principles underlying the design of the trigonometric scales appears in Chapter VII, §89.

37. Some important formulas from plane trigonometry. The following formulas from plane trigonometry, given for the convenience of the student, will be employed in the slide rule solution of trigonometric problems considered in this and the following chapter.

In the right triangle ABC of Fig. 14, the side opposite the angle A is designated by a, the side opposite B by b, and the hypotenuse by c. Referring to this figure, we write the following definitions and relations:

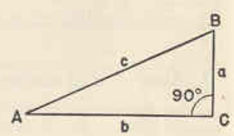


FIG. 14.

Definitions of the sine, cosine, and tangent:

$$\text{sine } A \text{ (written sin } A) = \frac{a}{c} = \frac{\text{opposite side}}{\text{hypotenuse}}, \qquad (1)$$

$$\text{cosine } A \text{ (written cos } A) = \frac{b}{c} = \frac{\text{adjacent side}}{\text{hypotenuse}}, \qquad (2)$$

$$\text{tangent } A \text{ (written tan } A) = \frac{a}{b} = \frac{\text{opposite side}}{\text{adjacent side}}. \qquad (3)$$

Reciprocal relations:

$$\text{cosecant } A \text{ (written csc } A) = \frac{c}{a} = \frac{1}{\sin A}, \qquad (4)$$

$$\text{secant } A \text{ (written sec } A) = \frac{c}{b} = \frac{1}{\cos A}, \qquad (5)$$

$$\text{cotangent } A \text{ (written cot } A) = \frac{b}{a} = \frac{1}{\tan A}. \qquad (6)$$

Relations between complementary angles:

$$\sin A = \cos (90° - A), \qquad (7)$$
$$\cos A = \sin (90° - A), \qquad (8)$$
$$\tan A = \cot (90° - A), \qquad (9)$$
$$\cot A = \tan (90° - A). \qquad (10)$$

Relations between supplementary angles:

$$\sin (180° - A) = \sin A, \qquad (11)$$
$$\cos (180° - A) = -\cos A, \qquad (12)$$
$$\tan (180° - A) = -\tan A. \qquad (13)$$

Relations between angles in a right triangle:

$$A + B = 90°. \qquad (14)$$

38. The *S* (Sine) and *SRT* (Sine, Radian, Tangent) scales. The graduations on the sine scales *S* and *SRT* represent *angles*. Accordingly, for convenience, we shall speak of *pushing the hairline to an angle* or *drawing an angle under the hairline*.

The *S* scale serves a double function. When read from left to right, using the *black* numbers, it covers the angles from 5.5° to 90°, and is used for finding *sines*. When read from right to left, using the *red* numbers, it covers the angles from 0° to 84.5°, and is used for finding *cosines*. The sine scale is the predominant scale. In what follows, any reference to an angle on a trigonometric scale will be the angle in black unless otherwise indicated.

The *SRT* scale covers the angles from 0.55° to 6°, and is used for finding sines, radian equivalents, and tangents of these small angles. This article deals primarily with sines; the use of the *SRT* scale for radians and tangents is covered in §40 and §41.

Note that the *S* and *SRT* scales are essentially one continuous scale (with a slight overlap), read against two continuous cycles of the *C* scale. Fig. 15 represents this relationship.

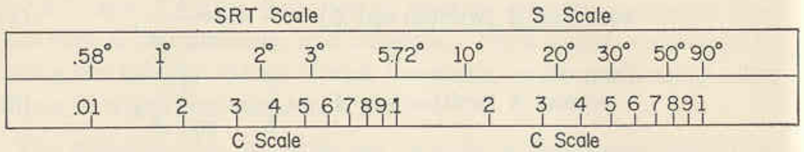


FIG. 15.

In order to set the hairline to an angle on the *S* scale, it is necessary to determine the values of the angles represented by the subdivisions. Since there are ten primary intervals between 8° and 9° each represents 0.1°; since each of the primary intervals is subdivided into two secondary intervals each of the latter represents 0.05°. Again since there are five primary intervals between 20° and 25°, each represents 1°; since each primary interval here is subdivided into five secondary intervals each of the latter represents 0.2°. The last mark at the right end represents 90°, the next mark to the left 85°, and the third 80°.

Rule.* *When the hairline is set to an angle on the S or the SRT scale, the sine of the angle is on scale C at the hairline, and hence on scale D when the rule is closed. Also when the hairline is set to an angle on the cosine scale (S red) the cosine of the angle is on scale C at the hairline.*

Each small inscription at the right end of a scale is called the **legend** of the scale. A legend of a scale specifies a range of values associated with the function represented by the scale. Thus *the legend* **0.1 to 1.0** *of scale S specifies that the sines of the angles on S and the cosines of angles on S red range from 0.1 to 1, and the legend* **0.01 to 0.1** *of the SRT scale indicates that sines (or radian equivalents and tangents) of angles on SRT range from 0.01 to 0.1.*

On the DECI-LON slide rule, the *S*, *SRT*, and *T* scales are extended slightly beyond the left hand index of the slide, primarily for convenience in reading angles near this end of the scale. The positions of the trigonometric scales opposite the left and right indexes of the

* See Appendix A for explanation of the ′ and ″ marks for finding sines of angles expressed in minutes or seconds.

C scale will be referred to as the left and right indexes of these scales. When referring to angles on each of these scales, we will mean the angles between the indexes.

Example. Evaluate (*a*) sin 36.4°, (*b*) sin 3.40°.

Solution. (*a*) Opposite 36.4° on *S*,
read 593 on *C* (or *D* when rule is closed).

To locate the decimal point, we note that the legend on the *S* scale indicates that resulting values must lie between 0.1 and 1.0. Hence the answer is **0.593.**

Solution. (*b*) Opposite 3.40° on *SRT*,
read 593 on *C*.

To locate the decimal point, we observe that the legend on the *SRT* scale establishes values as lying between 0.01 and 0.1. Therefore the final answer is **0.0593.**

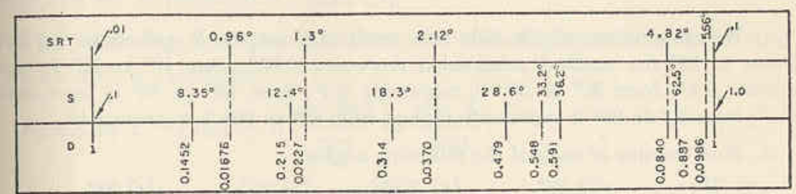


FIG. 16.

Fig. 16 shows scales *SRT*, *S*, and *D* on which certain angles and their sines are indicated. As an exercise close your slide rule and read the sines of the angles shown in the figure and compare your results with those given. Note that the values of sines appearing in Fig. 16 conform with the corresponding legends.

Each angle on *S red* is 90° minus the corresponding angle on *S black*. Also, equations (7) and (8) §37 are

$$\sin A = \cos (90° - A), \cos A = \sin (90° - A).$$

Hence, when the hairline is set to an angle *A* on *S*, it is set to sin *A* and to cos (90° − *A*) on scale *C*. For example

set the hairline to 25° on *S*,
at the hairline read on *C* **0.423** = sin 25° = cos 65°.

To find the cosine of an angle greater than 84.5°, use cos A = sin (90° − A). Thus to find cosine 86.9°, write cosine 86.9° = sin 3.1° and opposite 3.1° on SRT read on C **0.0541** *= sin 3.1° = cos 86.9°.*

The following is a tabular summary of the rules for finding the values of sines and cosines:

SINES

Angles from 0.573° to 5.73°:	Use *SRT* scale; values will lie between 0.01 and 0.10.
Angles from 5.73° to 90°:	Use *S* scale, black numbers, reading left to right; values will lie between 0.1 and 1.0.

COSINES

Angles from 0° to 84.25°:	Use *S* scale, red numbers, reading right to left; values will lie between 0.1 and 1.0.
Angles from 84.25° to 89.427°:	Use *SRT* scale to find the sine of 90° minus the angle; values will lie between 0.01 and 0.10.

EXERCISES

1. By examination of the slide rule verify that on the *S* scale from the left index to 10° the smallest subdivision represents 0.05°; from 10° to 20° it represents 0.1°; from 20° to 30° it represents 0.2°; from 30° to 60° it represents 0.5°; from 60° to 80° it represents 1°; and from 80° to 90° it represents 5°.

2. Find the sine of each of the following angles:

(*a*) 30°. (*b*) 38°. (*c*) 3.33°. (*d*) 90°. (*e*) 88°.
(*f*) 1.583°. (*g*) 14.63°. (*h*) 22.4°. (*i*) 11.80°. (*j*) 51.5°.

3. Find the cosine of each of the angles in Exercise 2.

4. Find x in each equation:

(*a*) $\sin x = 0.5$. (*d*) $\sin x = 0.1$. (*g*) $\sin x = 0.062$.
(*b*) $\sin x = 0.875$. (*e*) $\sin x = 0.015$. (*h*) $\sin x = 0.031$.
(*c*) $\sin x = 0.375$. (*f*) $\sin x = 0.62$. (*i*) $\sin x = 0.92$.

5. Find x in each equation:

(*a*) $\cos x = 0.5$. (*d*) $\cos x = 0.1$. (*g*) $\cos x = 0.062$.
(*b*) $\cos x = 0.875$. (*e*) $\cos x = 0.015$. (*h*) $\cos x = 0.031$.
(*c*) $\cos x = 0.375$. (*f*) $\cos x = 0.62$. (*i*) $\cos x = 0.92$.

39. Simple operations involving the *S* and *SRT* scales. If the reader will reflect that when the hairline is set to an angle A on scale S, it is also set to $\sin A$ on C, he can easily see that sines and cosines of angles can be used in combined operations and proportions by means of the S and SRT scales just as square roots and reciprocals were used in Chapter III by means of the B scale and the CI scale.

Thus to find 8 sin 40°,

opposite 8 on *D* set index of *C*,
opposite 40° on *S* read **5.14** on *D*.

The decimal point was placed after observing on the slide rule that sin 40° is approximately 0.6 and therefore that 8 sin 40° is approximately 8 × 0.6 = 4.8. The legend of the *S* scale **0.1 to 1.0** indicates that the approximate value of sin 40° is 0.6, a value between 0.1 and 1.0.

To find 8/cos 40°,

opposite 8 on *D* set 40° of *S red*,
opposite index of slide read **10.44** on *D*.

Here the decimal point was placed after observing on the slide rule that cos 40° is nearly 0.8 and therefore that 8/cos 40° is nearly equal to 8/0.8 = 10. Here again the legend **0.1 to 1.0** of *S* indicates that cos 40° is between 0.1 and 1.0.

The following examples illustrate the use of proportions involving trigonometric functions:

Example 1. Find A if $\dfrac{\sin 36°}{270} = \dfrac{\sin A}{320}$.

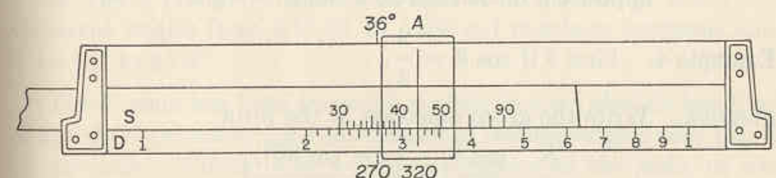


FIG. 17.

Solution (see Fig. 17). Here both parts in the first ratio are known. Hence write

$$\frac{S}{D}: \quad \frac{\sin 36°}{270} = \frac{\sin A}{320}, \text{ and}$$

opposite 270 on *D* set 36° of *S*,
push hairline to 320 on *D*,
at hairline read **44.2°** on *S*.

Example 2. Find A and x if $\dfrac{250}{\sin 32°} = \dfrac{330}{\sin A} = \dfrac{x}{\cos 80°}$.

Solution. Write

$$\frac{D}{S}:\ \frac{250}{\sin 32^\circ} = \frac{330}{\sin A} = \frac{x}{\cos 80^\circ};$$

opposite 250 on D set 32° of S,
push hairline to 330 on D,
at hairline read **44.40°** on S,
interchange indexes of slide,
push hairline to 80° on S *red*,
at hairline read **81.9** on D.

Here the decimal point was located by noting that $\sin 32^\circ = 0.5$ approx. and $\cos 80^\circ = 0.17$ approx. Hence

$$x = \frac{250 \times 0.17 \text{ approx.}}{0.5 \text{ approx.}} = 80 \text{ approx.}$$

Example 3. Find θ if $\sin\theta = \frac{3}{5}$.

Solution. Write the given equation in the form

$$\frac{S}{D}:\ \frac{\sin\theta}{3} = \frac{1\ (=\sin 90^\circ)}{5};$$

set right index of slide opposite 5 on D,
opposite 3 on D read **36.9°** on S.

Example 4. Find θ if $\cos\theta = \frac{2}{3}$.

Solution. Write the given equation in the form

$$\frac{S}{D}:\ \frac{\cos\theta}{2} = \frac{1\ (=\sin 90^\circ)}{3};$$

set right index of slide opposite 3 on D,
opposite 2 on D read **48.2°** on S *red*.

EXERCISES

1. In each of the following proportions find the unknowns:

(*a*) $\dfrac{\sin 50.4^\circ}{7} = \dfrac{\sin 42.2^\circ}{x} = \dfrac{\sin\theta}{8}$.

(*b*) $\dfrac{\sin\theta}{30.5} = \dfrac{\sin 35^\circ}{x} = \dfrac{\sin 60.5^\circ}{32.8}$.

(*c*) $\dfrac{\sin 25^\circ}{20} = \dfrac{\sin 40^\circ}{x} = \dfrac{\sin 70^\circ}{y}$.

(*d*) $\dfrac{\sin\theta}{15.6} = \dfrac{\sin\phi}{25.6} = \dfrac{\sin 12.92^\circ}{40.7}$.

2. Find the value of each of the following:

(*a*) $5\sin 30^\circ$. (*e*) $28\cos 25^\circ$.
(*b*) $12\sin 60^\circ$. (*f*) $35\csc 52.3^\circ$.
(*c*) $22/\sin 30^\circ$. (*g*) $17\sec 16^\circ$.
(*d*) $15/\sin 20^\circ$. (*h*) $55\sin 18^\circ$.

3. Find the value of θ in each of the following:

(*a*) $\sin\theta = \dfrac{307\sin 42.5^\circ}{2030}$. (*c*) $\sin\theta = \dfrac{433\sin 18.17^\circ}{136}$.

(*b*) $\sin\theta = \dfrac{413\sin 77.7^\circ}{488}$. (*d*) $\sin\theta = \dfrac{156\sin 12.92^\circ}{40.7}$.

4. Find the value of x in each of the following:

(*a*) $x = \dfrac{179.5\sin 6.42^\circ}{\sin 34.5}$. (*c*) $x = \dfrac{123.4\sin 8.20^\circ}{\sin 33.5^\circ}$.

(*b*) $x = \dfrac{3.27\sin 73^\circ}{\sin 2.22^\circ}$. (*d*) $x = \dfrac{375\sin 18.67^\circ}{\cos 62.7^\circ}$.

5. Find the value of x in each of the following:

(*a*) $x = \dfrac{4\sin 35^\circ - 5.4\sin 17^\circ}{\sin 47^\circ}$. (*c*) $x = \dfrac{18\sin 52.5^\circ - 23.4\cos 42.2^\circ}{\sin 22^\circ \sin 63^\circ}$.

(*b*) $x = \dfrac{8 - 6\sin 70^\circ}{\sin 37^\circ - 0.21}$. (*d*) $x = \dfrac{(27.7\sin 39.2^\circ)^2 - 16\cos 12.67^\circ}{46.2\sin 10.17^\circ + 32.1\sin 17.27^\circ}$.

---

40. The *T* (Tangent) scale. The black numbers on the T scale represent angles from 5.5° to 45°, the red numbers represent angles from 45° to 84.5°.

(The T scale has been extended slightly beyond the left hand index of the slide, primarily for convenience in reading angles near this end of the scale. Values lying within the indexes of the slide, to which the rules given in this article apply, are from 5.71° to 45° on T *black*, and from 45° to 84.29° on T *red*.)

Rule. *When the hairline is set to an angle A on T black, tan A is at the hairline on scale C, and hence on scale D when the rule is closed; when the hairline is set to an angle A on T red, tan A is at the hairline on CI (or on DI when rule is closed).*

Since $\tan 5.71^\circ = 0.1,\quad \tan 45^\circ = 1,\quad \tan 84.29^\circ = 10,$

the range of values on scale C for tangents of angles between 5.71° and 45° is 0.1 to 1, and on scale CI for tangents of angles between 45° and 84.29° is 1 to 10. The *black* legend **0.1 to 1.0** at the right end of the T scale indicates that tangents read on C *black* are between 0.1 and 1; the *red* legend **10.0 to 1.0** indicates that tangents read on *CI red* are between 10 and 1. The general rule governing the use of red and black numbers is given in the next article.

For example,

opposite 26° on *T black*, read **0.488** = tan 26° on *C*,
opposite 64° on *T red*, read **2.05** = tan 64° on *CI*.

The cotangent of an angle may be found by first using either of the identities (6) and (10) §37, namely

$$\cot A = 1/\tan A, \quad \cot A = \tan (90° - A)$$

to express the cotangent as the tangent of an angle and then using the method outlined above. Thus to find cot 26°, write from (10) $\cot 26° = \tan (90° - 26°) = \tan 64°$ and

opposite 64° on *T*, read **2.05** = cot 26° on *CI*,

or write from (6) $\cot 26° = 1/\tan 26°$ and

opposite 26° on *T*, read **2.05** = cot 26° on *CI*.

To find cot 64°, write $\cot 64° = \tan (90° - 64°) = \tan 26°$ and

opposite 26° on *T*, read **0.488** = cot 64° on *C*.

In computing an expression involving the tangent of an angle greater than 45° or any cotangent of an angle, it is advisable before beginning the computation to replace the tangent or cotangent by the tangent of an angle less than 45°. Thus to evaluate 565 tan 56° ÷ cot 42° we would first write

$$\frac{565 \tan 56°}{\cot 42°} = \frac{565 \cot 34°}{\cot 42°} = \frac{565 \tan 42°}{\tan 34°} \quad \text{and}$$

push the hairline to 565 on *D*,
draw 34° of *T* under the hairline,
push the hairline to 42° on *T*,
at the hairline read **754** on *D*.

The decimal point was placed after making the mental approximation $600 \times 0.9 \div 0.6 = 900$. The numbers 0.9 and 0.6 lie between 0.1 and 1.0, that is, within the range specified by the legend **0.1 to 1.0** of *T*.

It is shown in trigonometry that the sine and the tangent of an angle less than 5.71° are so nearly equal that they may be considered identical for slide rule purposes. Thus to find tan 2.25° and cot 2.25°,

opposite 2.25° on *SRT* read on *C*, **0.0393** = tan 2.25°,
opposite 2.25° on *SRT* read on *CI*, **25.5** = 1/tan 2.25° = cot 2.25°.

The operator should be careful in finding an angle greater than 45° on the tangent scale from a ratio. Thus to find *A* where $\tan A = \frac{5.6}{3.1}$, it is essential that the setting be made as though $\tan (90° - A)$ were to be found. In this case

$$\tan (90° - A) = \cot A = \frac{3.1}{5.6}, \text{ or } \frac{\tan (90° - A)}{3.1} = \frac{1}{5.6}.$$

Hence

opposite 56 on *D*, set 1 (= tan 45°) of *T*,
opposite 31 on *D*, read 90° − *A* = **29°** on *T black*,
or opposite 31 on *D*, read *A* = **61°** on *T red*.

Note that the setting must be made as though 90° − *A*, *an angle less than* 45°, *were to be found.*

EXERCISES

1. Fill out the following table:

ψ	8.1°	27.25°	62.32°	1.017°	74.25°	87°	47.47°
tan ψ							
cot ψ							

2. The following numbers are tangents of angles. Find the angles:
(*a*) 0.24. (*b*) 0.785. (*c*) 0.92. (*d*) 0.54. (*e*) 0.059. (*f*) 0.082. (*g*) 0.432. (*h*) 0.043. (*i*) 0.0149. (*j*) 0.374. (*k*) 3.72. (*l*) 4.67. (*m*) 17.01. (*n*) 1.03. (*o*) 1.232.

3. The numbers in Exercise 2 are cotangents of angles. Find the angles.

4. Find the angle *x* from each equation:
(*a*) $\tan x = \frac{3.7}{6.8}$. (*b*) $\tan x = \frac{287}{642}$. (*c*) $\tan x = \frac{5.72}{2.86}$. (*d*) $\tan x = \frac{8.52}{6.73}$. (*e*) $\cot x = \frac{5}{6}$. (*f*) $\cot x = \frac{17.2}{143}$.

41. Radians; small angles.* A radian is an angular unit equal to $\left(\frac{180°}{\pi}\right)$, or 57.3° accurate to three figures. The *SRT* scale is a

* See Appendix A for use of the "R" locating mark for converting radians to degrees and vice versa.

C scale whose marks represent numbers of degrees ranging from 0.573° to 5.73° approximately. It is so designed that the following rule holds:

Rule. *When the hairline is set to an angle in degrees on the SRT scale, it is also set to the same angle in radians on the C* scale, provided the number on the C scale is prefixed by "0.0" as indicated by the legend* 0.01 *to* 0.1 *at the end of the SRT scale.*

For example, in accordance with the rule,

push hairline to 3.56° on SRT,
at hairline read 621 on C.

Therefore 3.56° = **0.0621 radian.**

Observe that if we multiply both members of the equation

$$3.56° = 0.0621 \text{ radian}$$

by 10, 10^2, $\frac{1}{10}$, and $\frac{1}{10^2}$ in succession, we get

$$(10)\ (3.56°) = (10)\ (0.0621), \text{ or } 35.6° = 0.621 \text{ radian},$$
$$(100)\ (3.56°) = (100)\ (0.0621), \text{ or } 356° = 6.21 \text{ radians},$$
$$\left(\frac{1}{10}\right)(3.56°) = \left(\frac{1}{10}\right)(0.0621), \text{ or } 0.356° = 0.00621 \text{ radian},$$
$$\left(\frac{1}{100}\right)(3.56°) = \left(\frac{1}{100}\right)(0.0621), \text{ or } 0.0356° = 0.000621 \text{ radian}.$$

In general for any integer k, positive or negative

$$10^k(3.56°) = 10^k 0.0621 \textit{ radian}.$$

Now using the rule in reverse,

push the hairline to 1176 on C,
at hairline read 0.674° on SRT,

and conclude that

$$0.01176 \text{ radian} = \mathbf{0.674°}.$$

Multiplying this through by 10^2, $\frac{1}{10}$, and 10^k in succession, we get

$$1.176 \text{ radians} = 67.4°,$$
$$0.001176 \text{ radian} = 0.0674°,$$
$$10^k\ (0.01176) \text{ radians} = 10^k\ (0.674°).$$

*Of course the D scale may be used instead of the C scale when the rule is closed.

For angles θ in radians, where θ is less than 0.1 radian (or 5.73°), the following relation holds

$$\theta \text{ radians} \approx \sin\theta \approx \tan\theta, \qquad (15)$$

where the symbol "≈" means *"approximately equals"*. In other words, *the value of an angle in radians found by means of the italicized rule is also its sine and its tangent to slide rule accuracy.**

For example:

push hairline to 3.84° on SRT,
at hairline read 670 on C.

Therefore, in accordance with the italicized rule,

$$\sin 3.84° \approx \tan 3.84° \approx 0.0670,$$

and, in agreement with equation (15),

$$\sin 0.384° \approx \tan 0.384° \approx 0.00670,$$
$$\sin 0.0384° \approx \tan 0.0384° \approx 0.000670, \text{ etc.}$$

By using the relations of §37, the italicized rule above, and (15), we can find the values of other trigonometric functions of small angles.

For example:

$$\cot 1.352° = 1/\tan 1.352° \approx 1/\sin 1.352° = \csc 1.352°. \text{ Hence}$$

push hairline to 1.352° on SRT,
at hairline read on C, 0.0236 ≈ sin 1.352°,
at hairline read on CI, 40.31 ≈ csc 1.352° ≈ cot 1.352°.

Also to find cos 88.76°, use (8) §37 to get

$$\cos 88.76° = \sin(90° - 88.76°) = \sin 1.24°,$$

push hairline to 1.24° on SRT,
at hairline read 0.0216 on C. Therefore

$$\cos 88.76° = \sin 1.24° \approx \mathbf{0.0216}.$$

Then without moving the slide,

at hairline read 462 on CI

and conclude that

$$\sec 88.76° = 1/\cos 88.76° \approx \mathbf{46.2},$$
$$\tan 88.76° = 1/\cot 88.76° \approx 1/\cos 88.76° \approx \mathbf{46.2}.$$

Before beginning the exercises, the student should use the slide rule, the italicized rule of this section and (15) to verify the following approximate equations:

* The greatest error inherent in formula (15) is at $\theta = 0.1$ radian; it is nearly +0.0001 for sin 0.1° and −0.00033 for tan 0.1.° These errors are comparable in magnitude with other errors occurring in slide rule computation.

(*a*) $1.272° = 0.0222$ radian.
(*b*) $12.72° = 0.222$ radian.
(*c*) 0.0531 radian $= 3.04°$.
(*d*) 5.31 radians $= 304°$.
(*e*) $\sin 2.86° = 0.0499$.
(*f*) $\sin 0.286° = 0.00499$.
(*g*) $\tan 0.286° = 0.00499$.
(*h*) $\csc 0.286° = 200$.
(*i*) $\cot 0.286° = 200$.
(*j*) $\sec 87.25° = 20.8$.

EXERCISES

1. Express in radians:
(*a*) 1.416°. (*b*) 0.833°. (*c*) 2.5°. (*d*) 2.67°.
2. Express in degrees:
(*a*) 0.01823 radian. (*b*) 0.0462 radian. (*c*) 0.0865 radian.
3. Express in radians:
(*a*) 3.59°. (*b*) 0.0359°. (*c*) 35.9°. (*d*) 359°.
4. Express in degrees:
(*a*) 0.0296 radian. (*b*) 0.296 radian. (*c*) 0.000296 radian.
5. Express in radians:
(*a*) 912°. (*b*) 435°. (*c*) 0.000314°. (*d*) 2900°.
6. Find sin 3.42°, tan 3.42°, csc 3.42°, cot 3.42°.
7. Find sin 0.056°, tan 0.056°, csc 0.056°, cot 0.056°.
8. Find cos 89.75°, sec 89.75°, tan 89.75°, cot 89.75°.
9. Express in degrees the following angles expressed in radians:
(*a*) $\frac{\pi}{3}$. (*b*) $\frac{3\pi}{4}$. (*c*) $\frac{\pi}{72}$. (*d*) $\frac{\pi}{180}$. (*e*) $\frac{5\pi}{6}$.

Hint: Since π radians $= 180°$, replace π by 180°. However to change $\frac{5\pi}{6}$ radians to degrees

opposite 6 on *DF* set 5 of *C*,
opposite index of *D* read 15 on *SRT*, and
$5\pi/6$ radians $= 150$ degrees.

10. Evaluate the following:
(*a*) $83 \sin 0.0144°$.
(*b*) $500 \tan 0.0097°$.
(*c*) $432 \sin 0.716°$.
(*d*) $\frac{\tan 0.2°}{0.0001745}$.
(*e*) $\frac{\sin 0.3°}{0.131}$.
(*f*) $\frac{8 \sec 88.25°}{4.72}$

42. Other functions on the *S* and *T* scales. Because of the reciprocal relations (4), (5), and (6) of §37, the complementary relations

(7), (8), (9) and (10) of §37, the fact that with each black number n representing an angle on scales S and T is a red number representing $90° - n$, and that the numbers on CI are reciprocals of their opposites on C, four functions or angles may be read at once when the hairline is set to an angle on S or T. To perceive this and a rather interesting color relation, let (B) represent black and (R) represent red, and

push the hairline to 30° (B) or 60° (R) on S,
at the hairline read $\sin 30°$ (B) $= 0.5$ (B) on C,
at the hairline read $\cos 60°$ (R) $= 0.5$ (B) on C,
at the hairline read $\csc 30°$ (B) $= 2$ (R) on CI,
at the hairline read $\sec 60°$ (R) $= 2$ (R) on CI.

Again push the hairline to 35° (B) or 55° (R) on T,

at the hairline read $\tan 35°$ (B) $= 0.700$ (B) on C,
at the hairline read $\cot 55°$ (R) $= 0.700$ (B) on C,
at the hairline read $\cot 35°$ (B) $= 1.428$ (R) on CI,
at the hairline read $\tan 55°$ (R) $= 1.428$ (R) on CI.

These two illustrations indicate that *whenever the value of a direct function (sin, tan, sec) is read, the colors of the angle and its function are the same; whenever the value of a co-function (cos, cot, csc) is read, the colors of the angle and its function are different.* In other words: *direct functions (sin, tan, sec) are read on like colors (black to black, or red to red); co-functions (cos, cot, csc) are read on opposite colors (black to red, or red to black).*

EXERCISES

Using the red numbers on the trigonometric scales, solve Exercises 3 and 5 of §38, and Exercises 1 and 3 of §40.

43. Combined operations. The method for evaluating expressions involving combined operations as stated in §§12, 20 and 21 applies without change when some of the numbers are trigonometric functions. This is illustrated in the following examples:

Example 1. Evaluate $\frac{4 \sin 38°}{\tan 42°}$.

Solution. Push hairline to 4 on D,
draw 42° of T under the hairline,
push hairline to 38° on S,
at the hairline read **2.735** on D.

Example 2. Evaluate $\dfrac{6.1\sqrt{17}\sin 72^\circ \tan 20^\circ}{2.2}$.

Solution. Use Rule **(C)** §20 to write $\dfrac{\sqrt{17}\sin 72^\circ \tan 20^\circ}{2.2\left(\dfrac{1}{6.1}\right)}$.

Push hairline to 17 on A *right*,
draw 22 of C under the hairline,
push hairline to 20° on T,
draw 61 of CI under the hairline,
interchange the indexes,
push hairline to 72° on S,
at the hairline read **3.96** on D.

Example 3. Evaluate $\dfrac{7.9\csc 17^\circ \cot 31^\circ \cos 41^\circ}{18\tan 48^\circ \sqrt{3.8}}$.

Solution. Replacing csc 17° by $\dfrac{1}{\sin 17^\circ}$, cot 31° by $\dfrac{1}{\tan 31^\circ}$, and tan 48° by $\dfrac{1}{\tan 42^\circ}$ and using Rule **(C)** §20, we obtain

$$\frac{\left(\dfrac{1}{18}\right) 7.9 \tan 42^\circ \cos 41^\circ}{\sqrt{3.8}\sin 17^\circ \tan 31^\circ}.$$

Push hairline to 79 on D,
draw 38° of B *left* under the hairline,
interchange indexes,
push hairline to 18 on CI,
draw 17° of S under the hairline,
push hairline to 42° on T,
draw 31° of T under the hairline,
push hairline to 41° on S *red*,
at the hairline read **0.871** on D.

The student could have avoided the use of red numbers by replacing in the given expression cos 41° by sin 49°.

The CF scale may often be used to avoid shifting the slide. In the process of evaluating a fraction consisting of a number of factors in the numerator over a number of factors in the denominator, the hairline may be pushed to a number of the numerator on the CF scale provided that a number of the denominator on the CF scale is drawn under the hairline later in the process, and conversely. In other words the CF scale may be used at any time for a multiplication (or division) if it is later used for a division (or multiplication).

Example 4. Evaluate $\dfrac{2.10 \times 2.54 \times \sqrt{45}}{\sin 70^\circ \times \tan 35^\circ \times 3.06}$.

Solution. Push hairline to 21 on D,
draw 70° of S under the hairline,
push hairline to 254 on CF,
draw 35° of T under the hairline,
push hairline to 45 on B *right*,
draw 306 of CF under the hairline,
opposite index of C, read **17.77** on D.

Note that the folded scale was used twice, once in the third setting and once in the sixth.

EXERCISES

Evaluate the following:

1. $\dfrac{18.6\sin 36^\circ}{\sin 21^\circ}$.
2. $\dfrac{32\sin 18^\circ}{27.5}$.
3. $\dfrac{4.2\tan 38^\circ}{\sin 45.5^\circ}$.
4. $\dfrac{34.3\sin 17^\circ}{\tan 22.5^\circ}$.
5. $\dfrac{13.1\cos 40^\circ}{\tan 35.2^\circ}$.
6. $\dfrac{17.2\cos 35^\circ}{\cot 50^\circ}$.
7. $\dfrac{7.8\csc 35.5^\circ}{\cot 21.4^\circ}$.
8. $\dfrac{63.1\sec 80^\circ}{\tan 55^\circ}$.
9. $\dfrac{\sin 18^\circ \tan 20^\circ}{3.7\tan 41^\circ \sin 31^\circ}$.
10. $\dfrac{\sin 62.4^\circ}{8.1\tan 22.3^\circ}$.
11. $3.14\sin 13.17^\circ \csc 32^\circ$.
12. $7.1\,\pi\sin 47.6^\circ$.
13. $\dfrac{0.61\csc 12.25^\circ}{\cot 35.3^\circ}$.
14. $\dfrac{1\sin 22.7^\circ}{\tan 28.2^\circ}$.
15. $\dfrac{3.1\sin 61.6^\circ \csc 15.30^\circ}{\cos 27.7^\circ \cot 20^\circ}$.

16. $\dfrac{13.1 \sin 3.12°}{\tan 30.2°}$.

17. $\dfrac{0.0037 \sin 49.8°}{\tan 2.10°}$.

18. $\dfrac{\sqrt{16.5} \sin 45.5°}{\sqrt{4.6}\ 41.2 \cot 71.2°}$.

19. $\dfrac{\sqrt[3]{6.1}\ 4.91}{\tan 13.23°}$.

20. $\dfrac{\sin 51.5°}{(39.1)(6.28)}$.

21. $\dfrac{\csc 49.5°}{(19.1)(7.61)\sqrt{69.4}}$.

22. $(48.1)(1.68) \sin 39°$.

23. $0.0121 \sin 81° \cot 41°$.

24. $\dfrac{1.01 \cos 71.2° \sin 15°}{\sqrt{4.81} \cos 27.2°}$.

25. Solve for the unknowns in the following equations:

(a) $\dfrac{\tan \theta}{27} = \dfrac{\tan \alpha}{49} = \dfrac{\tan 33.2°}{38}$.

(b) $y = \dfrac{\tan 24.2°}{6.15} = \dfrac{\tan \theta}{1.07}$.

(c) $y = (407 \cot 82.88°)^2$.

(d) $y = \dfrac{17.2}{\tan 34.2°}$.

(e) $y = \dfrac{84.1 \tan 75°}{27.4}$.

(f) $y = \dfrac{9.32 \tan 17°}{32.2}$.

(g) $y = \dfrac{10.7}{15.1 \cot 42°}$.

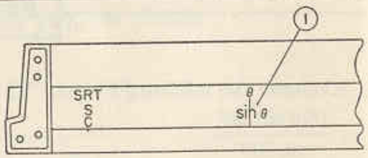
(h) $\tan \theta = \dfrac{4.77 \tan 21.2°}{25.7}$

(i) $\tan \theta = \dfrac{472 \tan 11.75°}{333}$.

44. Visual summary.

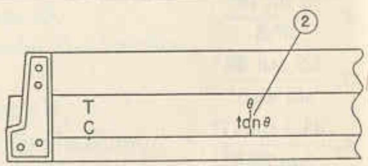
Sine of angle: $x = \sin \Theta$.

1. At θ on S (SRT) read $\sin \theta$ on C.



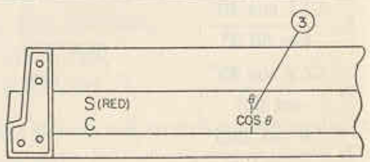
Tangent of angle: $x = \tan \Theta$.

2. At θ on T (black or red) read $\tan \theta$ on C (CI).



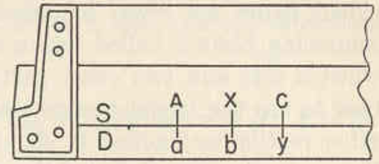
Cosine of angle: $x = \cos \Theta$.

3. At θ on S (red) read $\cos \theta$ on C.



Proportions involving trigonometric functions: $\dfrac{\sin A}{a} = \dfrac{\sin X}{b} = \dfrac{\sin C}{y}$.

1. To a on D set angle A on S,
2. at b on D read angle X on S,
3. at angle C on S read y on D.



CHAPTER V

SOLUTIONS OF TRIANGLES*

45. Introduction. When enough parts of a rectilinear (straight-sided) figure are given to determine it, the process of finding the remaining parts is called *solving the figure.* A triangle is determined when a side and two other parts are given. This chapter explains how to use the trigonometric scales in the solutions of triangles and other rectilinear figures. It should be noted that in these solutions, the *proportion principle* and the *sine law* are basic.

In addition to the solution of triangles, this chapter also includes a brief explanation of a method for solving practical problems involving triangles by the combined use of vectors and the trigonometric scales.

The following is a quick reference guide to the solutions of triangles explained in this chapter:

Given Data	See §
Angle, opposite side, any other part (See §52 for ambiguous case)	46
Two legs of right triangle	48, 49
Two sides and included angle	50
Three sides	51
Rectilinear figures	54
Vectors	55
Spherical triangles	56-58

46. Solution of a triangle for which an angle, the opposite side, and any other part are given. In the conventional way of lettering a triangle, each side is represented by a small letter and the opposite angle by the same letter capitalized. Thus, in Fig. 18, each of the pairs, a and A, b and B, c and C represents a side and the angle opposite. The law of sines is

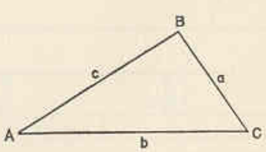


FIG. 18.

*For a more detailed treatment of the solution of triangles by logarithmic computation as well as by slide rule, see "Plane and Spherical Trigonometry" by Kells, Kern and Bland, McGraw-Hill Book Co., New York, 1942.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Using this law and the method of solving proportions explained in §39, we can solve any triangle for which a side, the opposite angle, and another part are given.

Example 1. Given a triangle (see Fig. 19) in which $a = 50$, $A = 65°$, and $B = 40°$, find b, c, and C.

Solution. Since $A + B + C = 180°$, $C = 180° - (A + B) = 75°$.

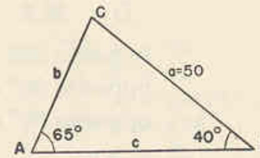


FIG. 19.

Application of the law of sines to the triangle gives

$$\frac{S}{D}\colon \frac{\sin 65°}{50} = \frac{\sin 40°}{b} = \frac{\sin 75°}{c}.$$

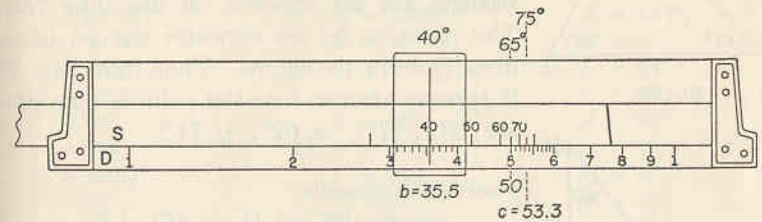


FIG. 20.

Accordingly (see Fig. 20),

opposite 50 on D set 65° of S,
push hairline to 40° on S,
at hairline read b = 35.5 on D,
push hairline to 75° on S,
at hairline read c = 53.3 on D.

Example 2. Find the unknown parts of the triangle (see Fig. 21) in which $a = 38.3$, $A = 25°$, $B = 38°$.

Solution. In this solution, it is necessary to use $\sin C = \sin 117°$. By (11) of §37, $\sin 117° = \sin (180° - 117°) = \sin 63°$. Hence we shall use sin 63° instead of sin 117° since the S scale does not provide directly for 117°. In general, use **the exterior angle of a triangle in the law of sines when the interior angle is greater than 90°.**

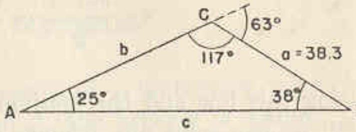


FIG. 21.

Hence from Fig. 21 write

$$\frac{S}{D}: \frac{\sin 25°}{38.3} = \frac{\sin 38°}{b} = \frac{\sin 63°}{c} \text{ and}$$

opposite 383 on D, set 25° of S,
opposite 38° on S, read b = **55.8** on D,
opposite 63° on S, read c = **80.7** on D.

47. Short cut in solving a triangle. Observe that it is not necessary to write the law of sines in solving a triangle. In accordance with the setting based on the law of sines, **opposite parts on a triangle are set opposite on the slide rule.** The parts to be set opposite can be taken directly from the figure. Thus from Fig. 22 it appears at once that the pairs of opposites are: 68.7, 47°; x, 62°; y, 71°.

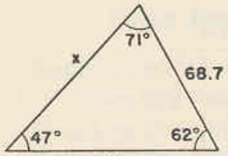


FIG. 22.

To solve the triangle,

opposite 687 on D, set 47° of S,
opposite 62° on S, read x = **82.9** on D,
opposite 71° on S, read y = **88.8** on D.

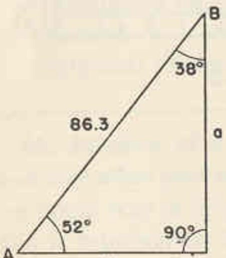


FIG. 23.

To solve the right triangle of Fig. 23, note that 90° and 86.3 are opposite and

opposite 863 on D, set 90° of S,
opposite 52° on S, read a = **68.0** on D,
opposite 38° on S, read b = **53.1** on D.

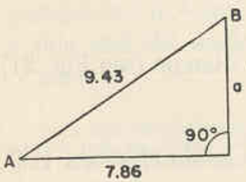


FIG. 24.

To solve the right triangle of Fig. 24,

opposite 943 on D, set 90° of S,
opposite 786 on D, read B = **56.5°** on S,
compute $A = 90° - B$ = **33.5°**,
opposite 33.5° on S, read a = **5.21** on D.

In general, *to solve any triangle for which a side and the angle opposite are known,*

opposite the known side on D set opposite angle of S,
opposite any known side on D read opposite angle on S,
opposite any known angle on S read opposite side on D.

EXERCISES

Solve the triangle having the given parts:

1. $a = 50$, $A = 65°$, $B = 40°$.
2. $c = 60$, $A = 50°$, $B = 75°$.
3. $a = 550$, $A = 10.2°$, $B = 46.6°$.
4. $a = 795$, $A = 79.98°$, $B = 44.68°$.
5. $a = 50.6$, $A = 38.67°$, $C = 90°$.
6. $a = 729$, $B = 68.83°$, $C = 90°$.
7. $b = 200$, $A = 64°$, $C = 90°$.
8. $c = 11.2$, $A = 43.5°$, $C = 90°$.
9. $b = 47.7$, $B = 62.93°$, $C = 90°$.
10. $a = 83.4$, $A = 72.12°$, $C = 90°$.
11. $a = 60$, $c = 100$, $C = 90°$.
12. $a = 0.624$, $c = 0.91$, $C = 90°$.
13. $b = 4250$, $A = 52.68°$, $C = 90°$.
14. $b = 2.89$, $c = 5.11$, $C = 90°$.
15. $b = 512$, $c = 900$, $C = 90°$.
16. $a = 52$, $c = 60$, $C = 90°$.
17. $a = 120$, $b = 80$, $A = 60°$.
18. $b = 91.1$, $c = 77$, $B = 51.1°$.

*19. $a = 50$, $c = 66$, $A = 123.2°$.

20. $a = 8.78$, $c = 10$, $A = 61.4°$.
21. $b = 0.234$, $c = 0.198$, $B = 109°$.

**22. $a = 21$, $A = 4.17°$, $B = 75°$.

***23. $b = 8$, $a = 120$, $A = 60°$.

24. $a = 40$, $b = 3$, $A = 75°$.
25. $c = 35.7$, $A = 58.65°$, $C = 90°$.
26. $c = 0.726$, $B = 10.85°$, $C = 90°$.
27. $a = 0.821$, $B = 21.57°$, $C = 90°$.

28. The length of a kite string is 250 yd., and the angle of elevation of the kite is 40°. If the line of the kite string is straight, find the height of the kite.

29. A vector is directed due N.E. and its magnitude is 10. Find the component in the direction of north.

30. Find the angle made by the diagonal of a cube with the diagonal of a face of the cube drawn from the same vertex.

* sin 123.2° = sin (180°-123.2°) = sin 56.8°.
** The *SRT* scale must be used for 4.17°.
*** The *SRT* scale must be used for angle *B*.

31. A ship at point S can be seen from each of two points, A and B, on the shore. If $AB = 800$ ft., angle $SAB = 67.7°$, and angle $SBA = 74.7°$, find the distance of the ship from A.

32. To determine the distance of an inaccessible tower A from a point B, a line BC and the angles ABC and BCA were measured and found to be 1000 yd., 44°, and 70°, respectively. Find the distance AB.

---

48. The law of sines applied to right triangles with two legs given. When the two legs of a right triangle are the given parts, we may first find the smaller acute angle from its tangent and then apply the law of sines to complete the solution.

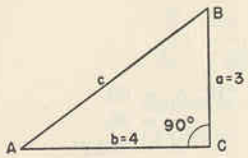


FIG. 25.

Example. Given the right triangle of Fig. 25 in which $a = 3$, $b = 4$, solve the triangle.

Solution. From the triangle we read $\tan A = \frac{3}{4}$. Hence write

$$\frac{T}{D}\colon\ \frac{\tan A}{3} = \frac{1}{4},\ \text{and}$$

opposite 4 on D set right index of C,
push hairline to 3 on D (Fig. 26a),
at hairline read A = **36.88°** on T,
at hairline read B = **53.12°** on *T red.*

Now complete the solution by using the method of §46. Since the hairline is set to 3 on D, draw the opposite angle 36.88° of S under the hairline (Fig. 26b), and opposite 1 (=sin 90°) on S read c=5 on D.

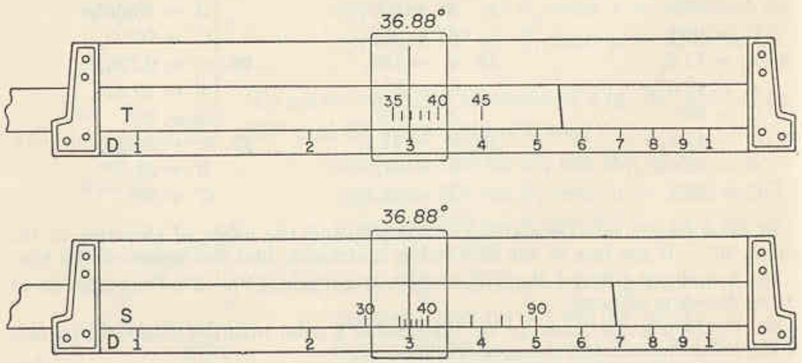


FIG. 26.

The following rule is based on the solution just completed. Those operators who have occasion to solve many right triangles of the type under consideration should use it.

Rule. *To solve a right triangle for which two legs are given,*
to larger leg on D set proper index of slide,
push hairline to smaller leg on D,
at the hairline read either angle (black or red) on T,
draw this angle on S under the hairline,
at index of slide read hypotenuse on D.

The solution of the triangle of Fig. 27 in accordance with the rule is as follows:

to 862 on D set right index of C,
push hairline to 479 on D,
at hairline read B = **29.08°** on T,
draw 29.08° on S under the hairline,
at index of S, read c = **9.86** on D.
Therefore $A = 90° - B = 60.92°$.

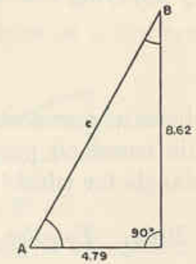


FIG. 27.

EXERCISES

Solve the following right triangles:

1. $a = 12.3$, $b = 20.2$.	4. $a = 273$, $b = 418$.	7. $a = 13.2$, $b = 13.2$.
2. $a = 101$, $b = 116$.	5. $a = 28$, $b = 34$.	8. $a = 42$, $b = 71$.
3. $a = 50$, $b = 23.3$.	6. $a = 12$, $b = 5$.	9. $a = 0.31$, $b = 4.8$.

10. The length of the shadow cast by a 10-ft. vertical stick on a horizontal plane is 8.39 ft. Find the angle of elevation of the sun.

---

49. Use of the *DI* scale in solving right triangles. The *DI* scale may be used to advantage in solving right triangles. For this purpose a useful proportion will be derived.

From Fig. 28 read

$$\frac{a}{c} = \sin A, \quad \text{or} \quad a = c \sin A;$$

$$\frac{a}{b} = \tan A, \quad \text{or} \quad a = b \tan A.$$

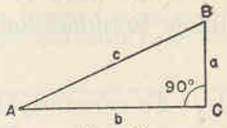


Fig. 28.

Equating these values of a obtain

$$a = b \tan A = c \sin A.$$

By applying rule B of §17, this may be written

$$\frac{1}{1/a} = \frac{\tan A}{1/b} = \frac{\sin A}{1/c}, \quad (1)$$

where *a represents the smaller leg of the triangle.* The following rule based on proportion (1) states a method of solving any right triangle for which two legs are known:

Rule. *To solve a right triangle when two legs are given:*

opposite smaller leg on DI set index of C,
opposite longer leg on DI read either angle (black or red) on T,
opposite this angle on S read hypotenuse on DI.

Example 1. Solve the right triangle having legs 3 and 4.

Solution. Setting $a = 3$, $b = 4$, in (1) obtain

$$\frac{1}{1/3} = \frac{\tan A}{1/4} = \frac{\sin A}{1/c}.$$

Opposite 3 on DI, set 1 (right) of C,
opposite 4 on DI, read $A = 36.9°$ on T,
opposite 36.9° on S, read $c = 5$ on DI,
$B = 90° - A = 53.1°$.

Example 2. Solve the right triangle having $a = 15$, $b = 8$.

Solution. Using the italicized rule,

opposite 8 on DI set 1 (left) of C,
opposite 15 on DI read $B = 28.1°$ on T,
opposite 28.1° on S read $c = 17$ on DI,
$A = 90° - B = 61.9°$.

EXERCISES

Use the method involving the DI scale to solve the following right triangles:

1. $a = 12.3$, $b = 20.2$.
2. $a = 101$, $b = 116$.
3. $a = 50$, $b = 23.3$.
4. $a = 273$, $b = 418$.
5. $a = 28$, $b = 34$.
6. $a = 12$, $b = 5$.
7. $a = 13.2$, $b = 13.2$.
8. $a = 42$, $b = 71$.
9. $a = 0.31$, $b = 4.8$.

10. The length of a shadow cast by a 10 ft. vertical stick on a horizontal plane is 8.39 ft. Find the angle of elevation of the sun.

11. The rectangular components of a vector r are 17.5 and 6.36 as shown in Fig. 29. Find the magnitude and direction of the vector.

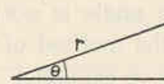


Fig. 29

12. Find the magnitude and direction of a vector having as the horizontal and vertical components 17.25 and 8.04, respectively.

50. Solution of a triangle for which two sides and the included angle are given. To solve an oblique triangle in which two sides and the included angle are given, it is convenient to divide the triangle into two right triangles. The method is illustrated in the following example.

Example. Given an oblique triangle in which $a = 6$, $b = 9$, and $C = 32°$, solve the triangle.

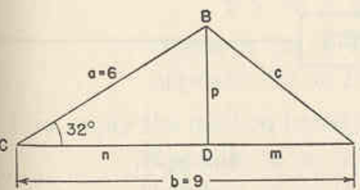


Fig. 30.

Solution. From B of Fig. 30, drop the perpendicular p to side b. Applying the law of sines to the right triangle CBD, we obtain

$$\frac{\sin 90°}{6} = \frac{\sin 32°}{p} = \frac{\sin 58°}{n}.$$

Solving this proportion, we find $p = 3.18$ and $n = 5.09$. From the figure, $m = 9 - 5.09 = 3.91$. Hence, in triangle ABD, we have

$$\tan A = \frac{p}{m} = \frac{3.18}{3.91}, \text{ or}$$

$$\frac{\tan A}{3.18} = \frac{1}{3.91}.$$

Solving this proportion, we get $A = 39.1°$. Now applying the law of sines to triangle ABD, we obtain

$$\frac{\sin 39.1°}{3.18} = \frac{\sin 90°}{c}.$$

Solving this proportion, we find $c = \mathbf{5.04}$. Finally, using the relation $A + B + C = 180°$, we obtain $B = \mathbf{108.9°}$. The italicized rule of §48 could have been used in place of the last two proportions.

If the given angle is obtuse, the perpendicular falls outside the triangle, but the method of solution is essentially the same as that used in the above example.

The law of cosines,

$$a^2 = b^2 + c^2 - 2bc \cos A,$$

may also be used for the solution. To solve the triangle of Fig. 30, we have

$$c^2 = a^2 + b^2 - 2ab \cos C, \text{ or}$$

$$c^2 = 6^2 + 9^2 - 2 \times 6 \times 9 \cos 32° = 36 + 81 - 91.6 = 25.4$$

and $c = \mathbf{5.04}$. Now using the setting based on the law of sines,

opposite 504 on D, draw 32° of S,
opposite 6 on D, read $A = \mathbf{39.1°}$ on S,
$B = 180° - 32° - 39.1° = \mathbf{108.9°}$.

The solution is checked by pushing the hairline to 71.1° (=180° − 108.9°) on S and reading 9 on D at the hairline.

EXERCISES

Solve the following triangles:

1. $a = 94$, $b = 56$, $C = 29°$.
2. $a = 100$, $c = 130$, $B = 51.8°$
3. $a = 235$, $b = 185$, $C = 84.6°$.
4. $b = 2.30$, $c = 3.57$, $A = 62°$.
5. $a = 27$, $c = 15$, $B = 46°$.
6. $a = 6.75$, $c = 1.04$, $B = 127.2°$.

7. $a = 0.085$, $c = 0.0042$, $B = 56.5°$.
8. $a = 17$, $b = 12$, $C = 59.3°$.
9. $b = 2580$, $c = 5290$, $A = 138.3°$.

10. Solve exercises 1 to 5 by using the law of cosines to get the third side and then the law of sines to get the unknown angles.

11. The two diagonals of a parallelogram are 10 and 12 and they form an angle of 49.3°. Find the length of each side.

12. Two ships start from the same point at the same instant. One sails due north at the rate of 10.44 mi. per hr., and the other due northeast at the rate of 7.71 mi. per hr. How far apart are they at the end of 40 minutes?

13. In a land survey find the latitude and departure of a course whose length is 525 ft. and bearing N65.5° E. (See Fig. 31).

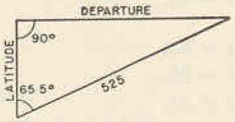


FIG. 31.

51. Solution of a triangle for which three sides are given. When the three sides are the given parts of an oblique triangle, we may find one angle by means of the law of cosines,

$$a^2 = b^2 + c^2 - 2bc \cos A,$$

and then complete the solution by using the law of sines.

Example. Given the oblique triangle of Fig. 32, in which $a = 15$, $b = 18$, and $c = 20$, find A, B, and C.

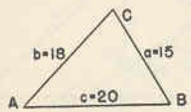


FIG. 32.

Solution. From the law of cosines we write $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$, or

$$\cos A = \frac{18^2 + 20^2 - 15^2}{2 \times 18 \times 20} = \frac{499}{720}, \text{ or } \frac{\cos A}{499} = \frac{1}{720}.$$

Opposite 720 on D, set right index of C,
opposite 499 on D, read $A = \mathbf{46.1°}$ on S *red.*

Now apply the method based on the law of sines and

opposite 15 on D, set 46.1° of S,
opposite 18 on D, read $B = \mathbf{59.9°}$ on S,
opposite 20 on D, read $C = \mathbf{74.0°}$ on S.

The relation $A + B + C = \mathbf{46.1° + 59.9° + 74.0°} = 180°$ serves as a check.

EXERCISES

Solve the following triangles:

1. $a = 3.41$, $b = 2.60$, $c = 1.58$.
2. $a = 111$, $b = 145$, $c = 40$.
3. $a = 35$, $b = 38$, $c = 41$.
4. $a = 61.0$, $b = 49.2$, $c = 80.5$.
5. $a = 7.93$, $b = 5.08$, $c = 4.83$.
6. $a = 21$, $b = 24$, $c = 27$.
7. $a = 97.9$, $b = 106$, $c = 139$.
8. $a = 57.9$, $b = 50.1$, $c = 35.0$.
9. $a = 13$, $b = 14$, $c = 15$.

10. The sides of a triangular field measure 224 ft., 245 ft., and 265 ft. Find the angles at the vertices.

11. Find the largest angle of the triangle whose sides are 13, 14, 16.

12. Solve Ex. 11 by means of the following formula:

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \text{ where } s = \frac{1}{2}(a+b+c).$$

13. In triangle ABC of Fig. 33 $p^2 = b^2 - m^2 = a^2 - n^2$. Hence $b^2 - a^2 = m^2 - n^2$. Factoring and replacing $(m+n)$ by c, we have

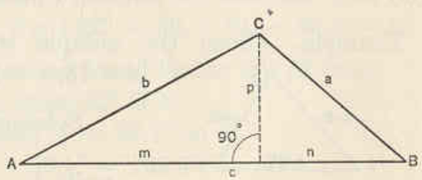


FIG. 33.

$$(b+a)(b-a) = (m+n)(m-n) = c(m-n), \text{ or}$$

$$\frac{b+a}{c} = \frac{m-n}{b-a}. \qquad (a)$$

To solve the triangle ABC, find $m - n$ by using proportion (a). Combine this result with

$$m + n = c,$$

to find m and n. Then solve each of the right triangles of triangle ABC and use the results to find the angles A, B, and C.

Apply this method to solve Exs. 1, 2, 3.

14. Another method of solving for angle A when sides a, b, and c are given follows. From the law of cosines, get

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{2bc + (b^2 - 2bc + c^2) - a^2}{2bc}$$

$$= 1 + \frac{(b-c)^2 - a^2}{2bc} = 1 - \frac{a^2 - (b-c)^2}{2bc}, \text{ or}$$

$$\cos A = 1 - \frac{(a-b+c)(a+b-c)}{2bc}. \qquad (b)$$

Thus to solve the triangle in which $a = 21$, $b = 24$, $c = 27$ for A, substitute these numbers in (b) to obtain

$$\cos A = 1 - \frac{(21 - 24 + 27)(21 + 24 - 27)}{2(24)(27)} = \frac{2}{3}, \text{ or } \frac{\cos A}{2} = \frac{1}{3}.$$

Opposite 3 on D, set right index of S,
opposite 2 on D, read $A = 48.2°$ on S *red.*

Now use the law of sines to get $B = 58.4°$, $C = 73.4°$.

Show that if $a = 97.9$, $b = 106$, $c = 139$, angle $A = 44.6°$.

Also use the method of this exercise to obtain angle A in Exercises 3, 4, 8, and 9.

---

52. Solution of a triangle for which two sides and an angle opposite one of them are given; the ambiguous case. When the given parts of a triangle are two sides and an angle opposite one of them, and when the side opposite the given angle is less than the other given side, there may be two triangles which have the given parts. We have already solved triangles in which the side opposite the given angle is greater than the other side. In this case there is always only one solution. Consider now a case where there are two solutions.

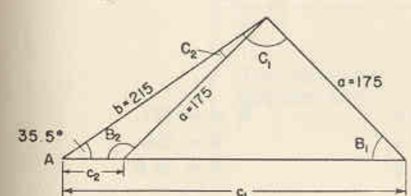


FIG. 34.

Example. Given $a = 175$, $b = 215$, and $A = 35.5°$; solve the triangle.

Solution. Fig. 34 shows the two possible triangles, AB_1C and AB_2C, having the given parts. To solve these triangles

opposite 175 on D, set 35.5° of S,
opposite 215 of D, read $B_1 = 45.5°$ on S,
$C_1 = 180° - A - B_1 = 99°$,
opposite 180° − 99° (=81°) read $c_1 = 298$ on S.

From Fig. 34 it appears that $B_2 = 180° - B_1 = 134.5°$. $C_2 = 180° - A - B_2 = B_1 - A = 10°$. Since 175 of D is opposite 35.5° of S, push hairline to 10° on S and read $c_2 = 52.3$ on D at the hairline.

It is instructive to observe that the slide was set only once, and that the required parts were obtained by pushing the hairline to parts already found and reading unknown parts at the hairline.

Let the three known parts of a triangle be a, b, and A. Fig. 35 represents the triangle with the given parts encircled. If a is less than b but greater than p, there are two triangles AB_1C and AB_2C having the given parts; if $a = p$ there is only one triangle ABC, and if a is less than p there will be no solution. Hence when p is found the operator knows the number of solutions to expect.

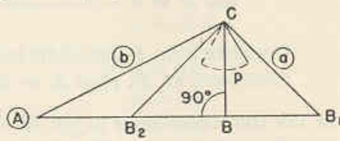


FIG. 35.

If a is greater than b, there will be one and only one triangle satisfying the given conditions.

EXERCISES

Solve the following oblique triangles:

1. $a = 18$, $b = 20$, $A = 55.4°$
2. $b = 19$, $c = 18$, $C = 15.82°$.
3. $a = 32.2$, $c = 27.1$, $C = 52.4°$.
4. $b = 5.16$, $c = 6.84$, $B = 44°$.
5. $a = 177$, $b = 216$, $A = 35.6°$.
6. $a = 17{,}060$, $b = 14{,}050$, $B = 40°$.

7. Find the length of the perpendicular p for the triangle of Fig. 36. How many solutions will there be for triangle ABC if (a) $b = 3$? (b) $b = 4$? (c) $b = p$?

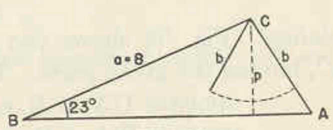


FIG. 36.

53. Methods of solving triangles; summary. The following tables summarize the methods of solution of triangles:

Type of Triangle	Known parts	Solve by
RIGHT	Any two parts other than two legs	Law of sines: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$
	Two legs	Find smaller acute angle from its tangent $\left(\frac{\text{smaller leg}}{\text{larger leg}}\right)$, then law of sines to find hypotenuse.
OBLIQUE	A side, opposite angle, and any other part	Law of sines: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$
	Two sides and included angle	Drop a perpendicular and solve the two right triangles thus formed; or find side opposite the given angle by using law of cosines: $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$
	Three sides	Find any angle by using law of cosines: $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ then use law of sines.

54. Rectilinear figures. The solutions of many practical problems are obtained by dealing with rectilinear figures. In finding the length of a specified line segment of a rectilinear figure, the beginner is likely to read a number of lengths which are not needed. This may be well at first, but the efficient operator reads and tabulates only useful numbers. The following examples and solutions indicate efficient methods of finding desired parts of rectilinear figures.

Example 1. Find the line segment marked x in Fig. 37.

Solution. By using the law of sines, we write

$$\frac{368}{\sin 39°} = \frac{y}{\sin 65°}, \quad \frac{y}{\sin 50°} = \frac{x}{\sin 28°}$$

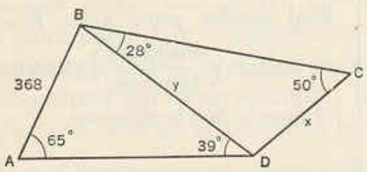


FIG. 37.

and then find x by making the following settings:
push hairline to 368 on D,
draw 39° of S under the hairline,
push hairline to 65° on S,
draw 50° of S under the hairline,
push hairline to 28° on S,
at the hairline read x = **325** on D.

The value of y was not tabulated, but it could have been read at the hairline on scale D when the hairline was set to 65° of scale S. Also it was not necessary to write the ratios; for, when one remembers that each ratio is that of a side of a triangle to the sine of the opposite angle, he has no difficulty in perceiving, from an inspection of the figure, the settings to be made.

Generally it is necessary to compute the magnitudes of a number of angles before the slide rule computation can be carried out. This process is illustrated in Example 2.

Example 2. Find the length of the side marked z in Fig. 38 (a).

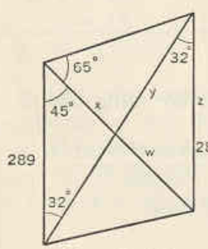


FIG. 38 (a).

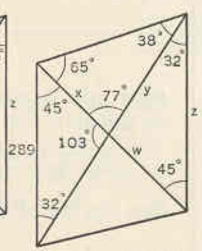


FIG 38 (b).

Solution. First draw Fig. 38 (b), compute the angles shown in the figure, and push the hairline to 289 on D, draw 38° of S under the hairline,
push hairline to 32° on S,
draw 45° of S under the hairline,
push hairline to 65° on S,
at the hairline read z = **319** on D.

In some problems it is necessary to perform some of the earlier settings in a chain of settings, compute some parts on the basis of the results, make some more settings, compute more parts, etc. This process is illustrated in Example 3.

Example 3. Find the side x of the inscribed quadrilateral shown in Fig. 39 (a).

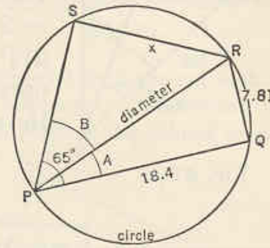


FIG. 39 (a).

Solution. Angles Q and S are right angles because each is inscribed in a semicircle. Knowing two legs of right triangle PQR, we first find its hypotenuse and then deal with triangle PSR. Accordingly
to 184 on D set left index of slide,
push hairline to 781 on D,
at the hairline read on T,
23° = A [Fig. 39(b)],
draw 23° of S under the hairline,
compute B [Fig. 39 (b)] = 65° − A = 42°,
interchange indexes
push hairline to 42° on S.
at the hairline read x = **13.37** on D.

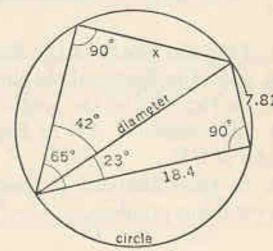


FIG. 39 (b).

The following example illustrates in more detail the same method of procedure.

Example 4. An engineer in a level country wishes to find the distance between two inaccessible points C and D and the direction of the line connecting them. He runs the line AB [Fig. 40 (a)] due North and measures the side and angles as indicated. Using his data, solve his problem.

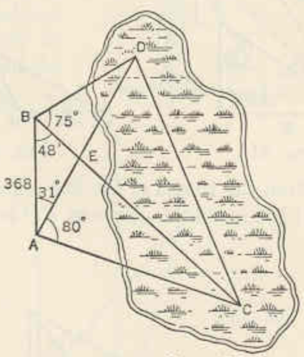


FIG. 40 (a).

Solution. First find EA (but do not write it), and then find EC = 766; afterwards find BE (but do not write it) and then ED = 425. In the triangle DEC [see Fig. 40 (b)] two sides and the included angle

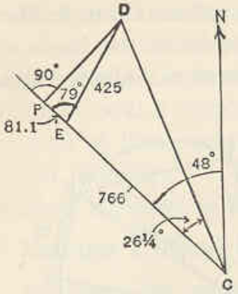


FIG. 40 (*b*).

are now known: hence the method of §49 may be applied to it to find $DC = 944$ and angle $ECD = 26\frac{1}{4}°$. Therefore the angle $NCD = 48° - 26\frac{1}{4}° = 21\frac{3}{4}°$, and line CD makes an angle of $21\frac{3}{4}°$ with a line directed due north. The operator may check these answers by making the suggested settings.

MISCELLANEOUS EXERCISES

1. Find the length of the line segment BC in Fig. 37.

2. Find the length of the line segment marked w in Fig. 38*a*.

3. In Fig. 41 find the length of the line segment marked x.

4. Line segment AB in Fig. 42 is horizontal and CD is vertical. Find the length of CD.

5. In the statement of Exercise 4, replace "Fig. 42" by "Fig. 43" and solve the resulting problem.

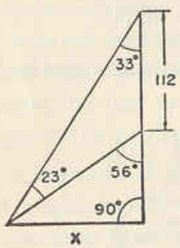


FIG. 41.

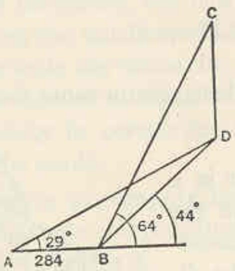


FIG. 42.

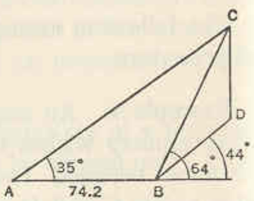


FIG. 43.

6. Find the length of the line segment marked x in Fig. 44.

7. If in Fig. 45 line segment BD is perpendicular to plane ABC, find its length.

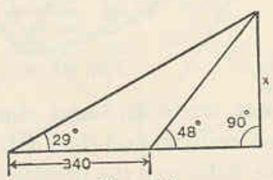


FIG. 44.

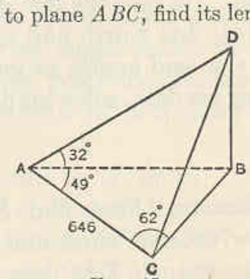


FIG. 45.

8. A tower and a monument stand on a level plane (see Fig. 46). The angles of depression of the top and bottom of the monument viewed from the top of the tower are 13° and 31° respectively; the height of the tower is 145 ft. Find the height of the monument.

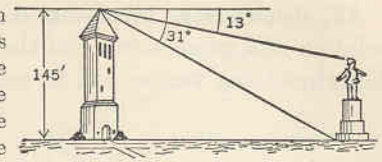


FIG. 46.

9. The captive balloon shown in Fig. 47 is connected to a ground station A by a cable of length 842 ft. inclined 65° to the horizontal. In a vertical plane with the balloon and its station and on the opposite side of the balloon from A a target B was sighted from the balloon on a level with A. If the angle of depression of the target from the balloon is 4° find the distance from the target to a point C directly under the balloon.

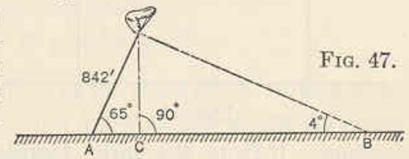


FIG. 47.

10. A lighthouse standing on the top of the cliff shown in Fig. 48 is observed from two boats A and B in a vertical plane through the lighthouse. The angle of elevation of the top of the lighthouse viewed from B is 16° and the angles of elevation of the top and bottom viewed from A are 40° and 23°, respectively. If the boats are 1320 ft. apart find the height of the lighthouse and the height of the cliff.

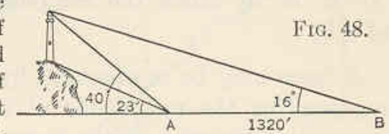


FIG. 48.

11. Fig. 49 represents a 600 ft. radio tower. AC and AD are two cables in the same vertical plane anchored at two points C and D on a level with the base of the tower. The angles made by the cables with the horizontal are 44° and 58° as indicated. Find the lengths of the cables and the distance between their anchor points.

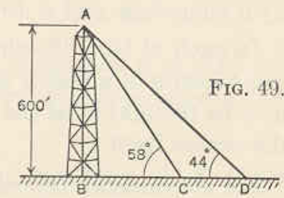


FIG. 49.

12. Two fixed objects, A and B of Fig. 50, were observed from a ship at point S to be on a straight line passing through S and bearing N 15° E. After sailing 5 miles on a course N 42° W the captain of the ship found that A bore due east and B bore N 40° E. Find the distance from A to B.

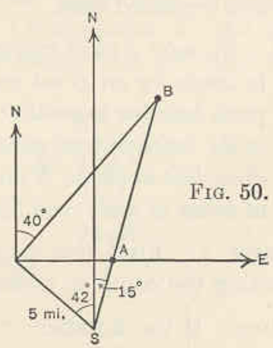


FIG. 50.

55. Applications involving vectors. Since vectors are used in the solution of a great number of the problems of science, a few applications involving vectors will be considered at this time.

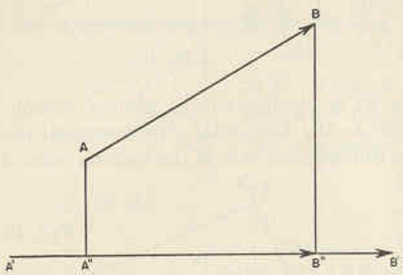


FIG. 51.

A vector AB (see Fig. 51) is a segment of a straight line containing an arrowhead pointed toward B to indicate a direction from its initial point A to its terminal point B. The length of the segment indicates the *magnitude* of the vector and the line with attached arrowhead indicates *direction*. If from the ends A and B of the vector, perpendiculars be dropped to the line of a vector $A'B'$ and meet it in the points A'' and B'', respectively, then the vector $A''\ B''$ directed from A'' to B'' is called the component of vector AB in the direction of $A'B'$.

A force may be represented by a vector, the length of the vector representing the magnitude of the force, and the direction of the vector the direction of the force. In fact, many quantities defined by a *magnitude* and a *direction* can be represented by vectors.

In each of the following applications, two mutually perpendicular components of a vector are considered. Evidently these components may be thought of as the legs of a right triangle having as hypotenuse the vector itself.

For convenience the rule for solving a right triangle when two legs are given is repeated here.

Rule. *To solve a right triangle for which two legs are given,*
to larger leg on D set proper index of slide,
push hairline to smaller leg on D,
at the hairline read either angle (black or red) on T,
draw this angle on S under the hairline,
at index of slide read hypotenuse on D.

Example 1. Find the magnitude and the angle of the vector representing the complex number $3.6 + j\,1.63$ where $j = \sqrt{-1}$.

Solution. If the numbers x and y be regarded as the rectangular coordinates of a point, the complex number $x + jy$ is represented by the vector from the origin to the point (x, y). Hence we must find R and θ in Fig. 52. Therefore, in accordance with the italicized rule stated above,

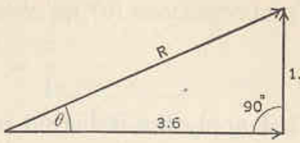


FIG. 52.

to 36 on D set right index of slide,
push hairline to 163 on D,
at the hairline read θ = **24.36°** on T,
draw 24.36° of S under the hairline,
at index of slide read R = **3.95** on D.

Example 2. A force of 26.8 lb. acts at an angle of 38° with a given direction. Find the component of the force in the given direction, and also the component in a direction perpendicular to the given one.

Solution. Denoting the required components by x and y (see Fig. 53), we write

$$\frac{26.8}{\sin 90^\circ} = \frac{y}{\sin 38^\circ} = \frac{x}{\sin 52^\circ},$$

make the corresponding setting, and read x = **21.1**, y = **16.5**.

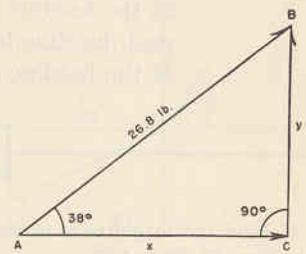


FIG. 53.

Example 3. A certain circuit consists of a resistance $R = 3.6$ and an inductive reactance $X = 2.7$ in series. Find the impedance z, the susceptance B, and the conductance G.

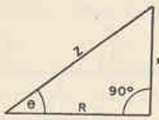


FIG. 54.

Solution. The quantities R, X and z have relations which may be read from Fig. 54. Conductance G and susceptance B are found from the relations

$$G = \frac{R}{R^2 + X^2}, \quad B = \frac{X}{R^2 + X^2},$$

or since $z = \sqrt{R^2 + X^2}$,

$$\left.\begin{aligned} G &= \frac{R}{\sqrt{R^2 + X^2}\,\sqrt{R^2 + X^2}} = \frac{\cos\theta}{z}, \\ B &= \frac{X}{\sqrt{R^2 + X^2}\sqrt{R^2 + X^2}} = \frac{\sin\theta}{z}. \end{aligned}\right\} \qquad (a)$$

From equations (a) we obtain

$$\frac{z}{1} = \frac{\sin\theta}{B} = \frac{\cos\theta}{G}. \qquad (b)$$

First apply the italicized rule stated above to find z and θ of Fig. 54, and then use the proportion principle to find B and G from (b). Hence

to 36 on D set right index of slide,
push hairline to 27 on D,
at the hairline read $\theta = 36.9°$ on T,
draw 36.9° on S under the hairline,
at index of slide read $z = 4.5$ on D,
draw 45 of C opposite left index of D,
push hairline to 36.9° on S,
at the hairline read $B = 0.1333$ on D,
push hairline to 36.9° *red* on S,
at the hairline read $G = 0.178$ on D.

EXERCISES

1. Find the magnitudes of the unknown vectors and of the unknown angles θ in Figs. 55, 56 and 57.

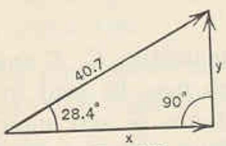


FIG. 55.

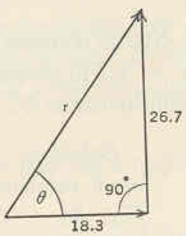


FIG. 56.

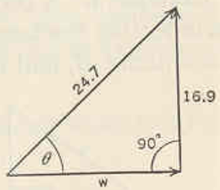


FIG. 57.

2. The rectangular components of a vector are 15.04 and 5.47 (see Fig. 58). Find the magnitude r and direction angle θ of the vector.

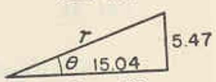


FIG. 58.

3. Find the magnitude and direction of a vector having as the horizontal and vertical components 18.12 and 8.45, respectively.

4. Find the horizontal and vertical components of a vector having magnitude 2.5 and making an angle of 16.25° with the horizontal.

5. A force of magnitude 28.8 lb. acts at an angle of 68° with the horizontal. Find its horizontal component, and its vertical component.

6. A 12-inch vector and an unknown vector r have as a resultant a 16-inch vector which makes an angle of 28° with the 12-inch vector as shown in Fig. 59. Find the unknown vector r.

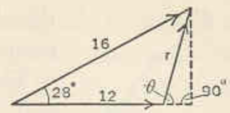


FIG. 59.

7. Find the magnitude and the angle of the vector representing the imaginary number $-2.7 + j3.6$. Hint: use Fig. 60.

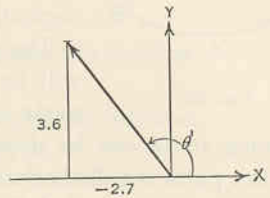


FIG. 60.

8. Through what angle θ measured counter-clockwise must a vector whose complex expression is $-10 - j5$ be rotated to bring it into coincidence with the vector whose complex expression is $3 + j4$? (See Fig. 61.)

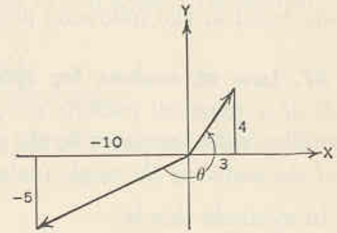


FIG. 61.

9. The complex expressions for two vectors (see Fig. 62) are $v_1 = 7 - j14$ and $v_2 = -6 - j8$. From the tip of v a line is drawn perpendicular to v_1. Find the length m of this perpendicular, and the length n of the line from the origin to the foot of the perpendicular.

10. A certain circuit consists of a resistance of 8.24 ohms and an inductive reactance of 4.2 ohms, in series. Find the impedance, the susceptance, and the conductance. (See Example 3.)

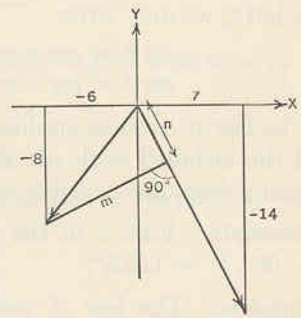


FIG. 62.

11. Find the impedance, the susceptance, and the conductance of a circuit which consists of a resistance of 8.76 ohms and an inductive reactance of 11.45 ohms in series.

56. Definitions relating to a sphere. A circle cut from a sphere by a plane through the center of the sphere is called a *great circle*.

The *arc length* of a great circle is generally stated as the angle subtended by the arc at the center of the sphere.

A *spherical triangle* consists of three arcs of great circles that form

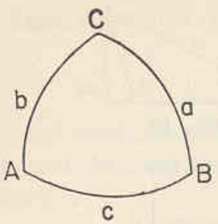


FIG. 63.

the boundaries of a portion of a spherical surface.

The vertices of the spherical triangle, Fig. 63, will be denoted by the capital letters A, B and C, and the sides opposite by a, b and c respectively.

The sides and angles of a spherical triangle will be referred to as its *parts*. When three parts of a spherical triangle are known the remaining parts can be determined. The process of finding these unknown parts is called *solving the triangle*.

Two great laws, called the *law of cosines* and the *law of sines*, form the basis for the solution of spherical triangles. These laws are considered in the following articles.

57. Law of cosines for spherical triangles. *The cosine of any side of a spherical triangle is equal to the product of the cosines of the two other sides increased by the product of the sines of the two other sides and the cosine of the angle included between them.*

In symbols this is:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

Similarly, we may write

$$\cos b = \cos a \cos c + \sin a \sin c \cos B;$$
$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

The law of cosines applies to a spherical triangle when two sides and the included angle are given. It is also used in the derivations of many formulas of spherical triangles.

Example. Find c in the spherical triangle for which $a = 76°$, $b = 58°$, $C = 116.5°$.

Solution. The law of cosines adapted to the given data yields $\cos c = \cos 76° \cos 58° + \sin 76° \sin 58° \cos 116.5°$. For convenience let $x = \cos 76° \cos 58°$ and $y = \sin 76° \sin 58° \cos 116.5°$. Hence $\cos c = x + y$. To find x make the following setting:

close the rule,
push hairline to 76° on *S red*,
draw right index of slide under the hairline,
push hairline to 58° on *S red*,
under hairline read on D, $0.1282 = x$.

The decimal point was determined by means of the legend of the S scale and a mental approximation. To obtain the value of y first note that $\cos 116.5° = -\cos (180° - 116.5°) = -\cos 63.5°$, write $y = \sin 76° \sin 58° (-\cos 63.5°)$, and make the following setting:

close the rule,
push hairline to 76° on S,
draw right index of slide under the hairline,
push hairline to 58° on S,
draw right index of slide under the hairline,
push hairline to 63.5° on *S red*,
under hairline read on D, $-.367 = y$.

The decimal point was determined as above.

Therefore $\cos c = x + y = 0.1282 - 0.367 = -0.2388$. To determine c:

close the rule,
push hairline to 2388 on D,
under hairline read 76.3° on *S red*.

Since $\cos c$ is negative, c must be in the second quadrant. Hence $c = 180° - 76.3° = 103.7°$.

EXERCISES

1. Using the law of cosines find the side opposite the given angle for a spherical triangle having:
   (a) $b = 60°$, $c = 30°$, $A = 45°$.
   (b) $a = 47°$, $c = 32°$, $B = 125°$.
   (c) $a = 48°$, $b = 64°$, $C = 120°$.

2. Use the law of cosines to find the indicated angle in the spherical triangle having:
   (a) $a = 60°$, $b = 60°$, $c = 60°$, $A = ?$
   (b) $a = 120°$, $b = 60°$, $c = 90°$, $B = ?$
   (c) $a = 75°$, $b = 35°$, $c = 60°$, $C = ?$

3. Use the law of cosines to find all the unknown parts of a spherical triangle having $a = 62°$, $b = 125°$, $C = 155°$.

4. The following formula is derived from the law of cosines:
   $\cos A = -\cos B \cos C + \sin B \sin C \cos a.$

Using this formula find the angle opposite the given side for a spherical triangle in which:
   (a) $B = 64°$, $C = 69°$, $a = 125°$.
   (b) $B = 135°$, $C = 125°$, $a = 140°$.

58. The law of sines for spherical triangles. If a, b and c represent the sides of a spherical triangle and A, B and C the respective opposite angles,

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

The above formula is known as the *law of sines.*

When a part of a spherical triangle is found using the law of sines, the part is either of the first quadrant or of the second quadrant. The following theorems from solid geometry will frequently make it possible to determine the quadrant:

Theorem I. *The order of magnitude of the sides of a spherical triangle is the same as the order of magnitude of the respective opposite angles*; or in symbols, if

$$a < b < c, \quad \text{then} \quad A < B < C$$

Theorem II. *The sum of two sides of a spherical triangle is greater than the third side.*

In many practical cases the quadrant of the required part is known beforehand. However, in cases where the quadrant cannot be determined by means of the above theorems or when the quadrant is not known beforehand, other formulas must be used.

Example. Use the law of sines to find b in the spherical triangle having $A = 130°$, $B = 32.5°$, $a = 84°$.

Solution. Apply the sine law to the given data to obtain

$$\frac{\sin b}{\sin 32.5°} = \frac{\sin 84°}{\sin 130°}.$$

Recalling that when the hairline is pushed to an angle on the S scale the sine of the angle is found under the hairline on the C scale (or *on the D scale when the rule is closed*), and also noting that when the hairline is set to a number on the D scale the angle whose sine is that number is under the hairline on the S scale, make the following setting in accordance with the proportion principle:

close the rule,
push hairline to 84° on S,
draw 50° (= 180° − 130°) of S under the hairline,
push hairline to 32.5° on S,
close the rule,
under hairline read **44.3°** on S.

The quadrant of b was determined by Theorem I.

In the above setting the proportion was solved essentially on the C and D scales. It should be noted that with the rule closed and the hairline pushed to 84° on the S scale, sin 84° is under the hairline on the D scale. One further observation is important. *In solving a proportion involving the sine law as applied to spherical triangles it is necessary to so make the setting that the unknown term appears on the D scale.* This is accomplished by always writing the sine law so that the unknown term appears in the numerator, then closing the rule and pushing the hairline to the number on S which represents the angle in the numerator of the known ratio of the sine law proportion.

EXERCISES

1. Use the law of sines to find c in the spherical triangle having

(a) $A = 70°$, $C = 95°$, $a = 58°$.
(b) $A = 70°$, $C = 131.3°$, $a = 57.9°$.

2. Check the following data by using the law of sines:

(a) $A = 47.4°$, $B = 22.3°$, $C = 146.7°$, $a = 116.8°$, $b = 27.4°$, $c = 138.3°$.
(b) $A = 110.2°$, $B = 133.3°$, $C = 70.3°$, $a = 147.1°$, $b = 155.1°$, $c = 33.1°$.

3. Use the law of cosines to find c and then the law of sines to find A and B in the spherical triangle having $a = 62°$, $b = 125°$, $C = 155°$.

4. Fig. 64 represents the so-called terrestrial triangle used in navigation. The parts symbolized on the figure are:

DL_0 meaning difference in longitude,
C_0L_1 meaning compliment of latitude of departure,
C_0L_2 meaning compliment of latitude of arrival,
d meaning great circle distance traveled*,
C meaning initial course angle to be steered.

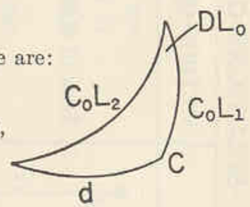


Fig. 64.

Substitution of these parts of the terrestrial triangle in the cosine law and the sine law yields the following formulas:

(1) $\cos d = \sin L_1 \sin L_2 + \cos L_1 \cos L_2 \cos DL_0$.

(2) $\dfrac{\sin C}{\cos L_2} = \dfrac{\sin DL_0}{\sin d}.$

Use formula (1) to find the great circle distance d, and then formula (2) to find the initial course angle C, for the indicated flights:

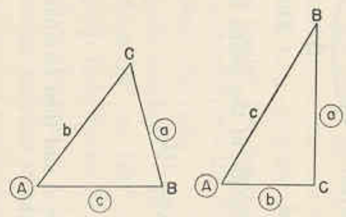
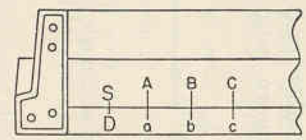
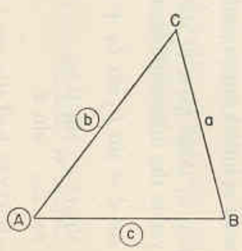
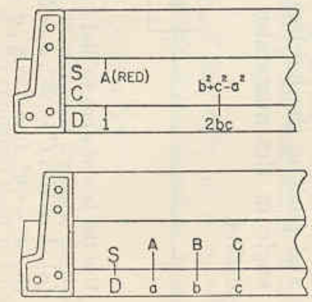
*1 minute of arc on a great circle of the earth is equal to 1 nautical mile.

(a) From Rome (L_1, 41.8° North; Longitude 12.3° East) to Dublin (L_2, 53.3° North; Longitude 6.3° West).

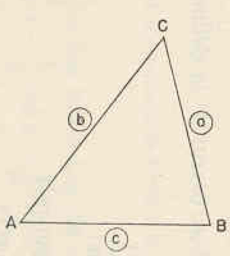
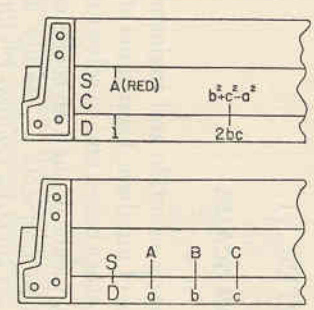
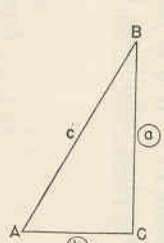
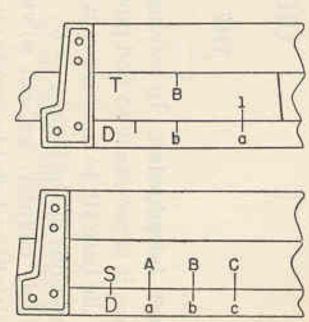
(b) From Tokyo (L_1, 35.7° North; Longitude 138.8° East) to Seoul (L_2, 37.6° North; Longitude 123.8° East).

59. Visual summary.

SOLUTIONS OF TRIANGLES

GIVEN PARTS (circled)	METHOD OF SOLUTION	SETTING
 An angle, the opposite side, and any other part	Law of sines: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$	
 Two sides and included angle	Find side opposite the given angle by using law of cosines: $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ Then use law of sines: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$	

SOLUTIONS OF TRIANGLES (Cont.)

GIVEN PARTS (circled)	METHOD OF SOLUTION	SETTING
 Three sides	Find any angle by using law of cosines: $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ Then use law of sines: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$	
 Right triangle: two legs	Find *smaller* angle by using $\tan B = \frac{b}{a}$ Then use law of sines to find other parts: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$	

CHAPTER VI

THE LON SCALES

60. Introduction. In solving many scientific and engineering problems, it is necessary to find powers of numbers, logarithms of numbers to the base 10, to the base e, or to other bases, and to solve exponential equations of the form $y = e^x$. By means of the Lon scales considered in this chapter, we can solve such problems as readily as we can multiply and divide simple numbers with the familiar C and D scales.

In Chapter VII the design of the Lon scales is discussed from a logarithmic viewpoint.

61. Laws of exponents; rules for combining logarithms. The following definitions and laws are taken from algebra and are here listed for review and easy reference.

Exponents

Definition I. If n is a positive integer,

$$a^n = a \cdot a \cdot a \cdots a \ (n \text{ factors}). \tag{1}$$

Definition II. If $a \neq 0$, $a^0 = 1$. (2)

Definition III. Whenever a has a principal nth root

$$a^{1/n} = \sqrt[n]{a}. \tag{3}$$

Definition IIIa. If a has a principal nth root, and m/n is a rational fraction in its lowest terms,

$$a^{m/n} = (a^{1/n})^m = (a^m)^{1/n} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}. \tag{4}$$

Definition IV. If q is positive, $a \neq 0$ and a^q is defined by one of the preceding definitions, then

$$a^{-q} = \frac{1}{a^q}. \tag{5}$$

Laws of Exponents

Law 1. $$a^p \cdot a^q = a^{p+q} \tag{6}$$

Law 1(a). $$(a^p)^q = (a^q)^p = a^{pq} \tag{7}$$

Law 1(b). $$(ab)^p = a^p b^p \tag{8}$$

Law 1(c). $$\left(\frac{a}{b}\right)^p = \frac{a^p}{b^p} \tag{9}$$

Law 2. $$\frac{a^p}{a^q} = a^{p-q} = \frac{1}{a^{q-p}} \tag{10}$$

Logarithms

Definition. Given the equation $a^y = x$, the solution for y is *by definition the logarithmic function*

$$y = \text{Log}_a\, x; \tag{11}$$

that is, y is the logarithm of x to the base a.

Rules for Combining Logarithms

I. $\text{Log}_a\, MN = \text{Log}_a\, M + \text{Log}_a\, N.$ (12)

II. $\text{Log}_a \dfrac{M}{N} = \text{Log}_a\, M - \text{Log}_a\, N.$ (13)

III. $\text{Log}_a\, M^p = p\, \text{Log}_a\, M.$ (14)

EXERCISES

Assume that literal numbers in this set of exercises are positive.

Simplify the expressions:

1. $(-2x^3y^{-5})^2$.
2. $(3x^4y^{-2})^{-4}$.
3. $\left(\dfrac{x^{-2}}{2y^{-3}}\right)^{-3}$.
4. $(2c^2d^{-3})^4$.
5. $\dfrac{(2c^{1/4}\, d^{-5/6})^2}{(-4^{3/4}\, d^{5/3})}$.
6. $[1 - (1 - x)^{-1}]^{-2}$.

In each of the equations from 7 to 10 transform the left member to the right member.

7. $\text{Log}\, \frac{5}{4} - \text{Log}\, 300 + \text{Log}\, 120 = -\,\text{Log}\, 2.$
8. $2\, \text{Log}\, x - \frac{1}{4}\, \text{Log}\, y - \frac{3}{4}\, \text{Log}\, x - \text{Log}\, y = \frac{5}{4}\, \text{Log} \left(\frac{x}{y}\right).$
9. $\text{Log}\, \sqrt[3]{x^2} + \text{Log}\, \sqrt{x^3} + \text{Log}\, \sqrt[12]{x} + \text{Log}\, 5\, \sqrt[4]{x^3} = \text{Log}\, 5x^3.$

62. The Lon scales. The eight scales labeled *Ln0, Ln1, Ln2, Ln3, Ln-0, Ln-1, Ln-2* and *Ln-3* will be referred to collectively as the "Lon scales," and any individual scale of the eight as a "Lon scale." The Lon scales are divided into two groups. The scales labeled *Ln0, Ln1, Ln2* and *Ln3* comprise one of these groups. They are all on the front face of the rule and are numbered in black. This group will be referred to as the "**black Lons,**" and the individual scales in the group as "Lon zero," "Lon one," etc. The group collectively exhibits a continuous sequence of numbers increasing from left to right, ranging from 1.001 to 30,000. Each scale of this group functions with the *C* and *D* scales on the front face of the rule.

The second group consists of the scales labeled *Ln-0, Ln-1, Ln-2* and *Ln-3*. This group of scales is on the reverse face of the rule and is numbered in red. We shall refer to this group as the "**red Lons,**" and the individual scales in the group as "Lon minus zero," "Lon minus one," etc. The group exhibits a continuum of decimal fractions increasing from right to left, ranging from 0.00003 at the right end of the Lon-minus-three scale to 0.999 at the left end of the Lon-minus-zero scale. These scales operate with the *C* and *D* scales on the reverse face of the slide rule.

Each calibration mark on the Lon scales represents a single unique number complete with decimal point. This is in contrast to the calibration marks on the *C*, *D* and other numerical scales, on which each mark may represent many numbers. For example, the calibration mark numbered 4 on the *C* scale may represent 4, 40, 400, 0.04, 0.004, etc., whereas the calibration mark numbered 4 on the *Ln3* scale represents 4 and no other number.

The left and right indexes of any Lon scale are defined to be the positions on that scale opposite the left and right indexes respectively of the *D* scale. For convenience of reading, the calibration marks on some Lon scales extend beyond the scale's indexes, that is, to the left of the left index and to the right of the right index. When referring to marks on any Lon scale, unless otherwise indicated, we will mean the marks between the indexes of the scale.

63. The number e. One of the most important and useful mathematical constants is the number e, defined by

$$e = \lim_{Z \to 0} (1 + Z)^{\frac{1}{Z}}$$

Like π, e is not a rational number. Its value is 2.71828 (to five decimal places). This constant is the base of the system of natural or Napierian logarithms, and it appears again and again in the mathematics of science, engineering and finance.

It is especially important in slide rule theory because it is basic in the design of the Lon scales.

64. Logarithms to the base e. In the equation $N = e^x$, the exponent x is defined to be the logarithm of N to the base e. This is called the natural or Napierian logarithm of N and is denoted by

$\text{Log}_e N$ or $\text{Ln}\, N$ (pronounced "Lon of N").

By using the Lon scales in conjunction with the *D* scale, Lns (logs to the base *e*) of numbers can readily be found, as well as logarithms to any base except 0 or 1.

The Lon scales are so designed that when the hairline is set to a number *N* on a Lon scale, Ln *N* is under the hairline on Scale *D*.

To use this fact in evaluating Ln 7.39 (see Fig. 65), push hairline to 7.39 on *Ln3*, and under hairline read 2.00 on *D*.

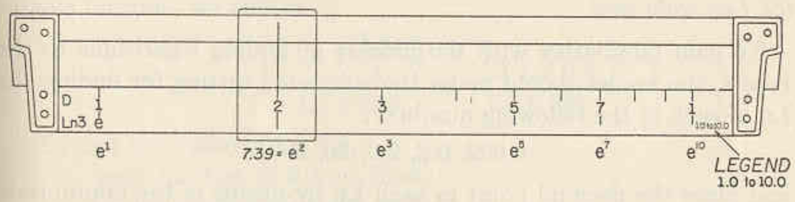


Fig. 65.

The position of the decimal point in the logarithm is determined in accord with the following facts:

The left index of the *Ln3* scale represents e^1, its Ln is 1; the right index represents e^{10}, its Ln is 10; hence the Lns of the numbers represented on *Ln3* range from **1 to 10.** Note that the legend at the right end of the *Ln3* scale is "**1.0 to 10.0.**" If Ln $N = x$, then the legend numbers **1.0 to 10.0** of *Ln3* are the limits of x when N is on *Ln3*. In like manner the legend of the *Ln-3* scale, **−1.0 to −10.0**, is based on the Lns of e^{-1} and e^{-10} respectively, represented by the indexes of the scale. Hence, the Lns of the numbers represented on *Ln-3* range from **−1.0 to −10.0.** The legend of each Lon scale has a similar relation to the numbers represented on the scale. In short, *the legend at the right end of each Lon scale gives the limits of the values of the Lns (read on D) of the numbers represented on that scale.*

In the previous example, 7.39 is found on *Ln3*, so the value of Ln 7.39, according to the legend of the *Ln3* scale, must lie between **1.0** and **10.0**. Hence, Ln 7.39 = **2**. This, of course, can be verified by a quick mental estimate.

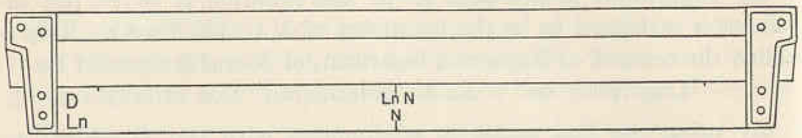


FIG. 66.

The rule for finding the logarithm to the base e of any number (Fig. 66) is:

Rule. *To find the value of Ln N, push the hairline to N on a Lon scale, read Ln N on the D scale under the hairline, and place the decimal point in this number so that the result lies between the legend numbers of the Lon scale used.*

To gain familiarity with the process of finding logarithms to the base e, the reader should make the suggested setting for finding the Ln of each of the following numbers:

0.002, 0.2, 2.0, 20, 200,

and place the decimal point in each Ln by means of the appropriate legend.

Opposite 0.002 on *Ln-3* read 621 on D, and in accord with the legend of *Ln-3* obtain Ln 0.002 = **−6.21**;
opposite 0.2 on *Ln-3* read 161 on D, and in accord with the legend of *Ln-3* obtain Ln 0.2 = **−1.61**;
opposite 2.0 on *Ln2* read 639 on D, and in accord with the legend of *Ln2* obtain Ln 2.0 = **0.693**;
opposite 20 on *Ln3* read 2995 on D, and in accord with the legend of *Ln3* obtain Ln 20 = **2.995**;
opposite 200 on *Ln3* read 530 on D, and in accord with the legend of *Ln3* obtain Ln 200 = **5.30**.

Example 1. Find Ln 1.345 and Ln 0.9946.

Solution. Push hairline to 1.345 on *Ln2*,
under hairline read on D **0.296** = Ln 1.345;
push hairline to 0.9946 on *Ln-0*,
under hairline read on D **−0.0054** = Ln 0.9946.

Here the decimal point was placed in each answer by observing that 0.296 lies within the legend range **0.1** to **1.0** of scale *Ln2*, the scale on which 1.345 is found, and that −0.0054 lies within the legend range **−0.001** to **−0.01** of scale *Ln-0*, the scale on which 0.9946 is found.

Example 2. If r is the per cent loss of a radioactive substance in n units of time, A_0 the initial amount of the substance, and A the amount present at any time t, the formula for t is

$$t = \frac{n \text{ Ln}\left(\frac{A}{A_0}\right)}{\text{Ln } (1\text{-r})}.$$

It is found that 0.5% of radium disappears in 12 years. Find the half life* of radium.

Solution. It is required to find t when $A = \frac{1}{2}A_0$. Here, $r = 0.5\% = 0.005$, $1\text{-}r = 0.995$, $n = 12$. Substituting these quantities in the above formula, we obtain

$$t = \frac{12 \text{ Ln}\left(\frac{\frac{1}{2}A_0}{A_0}\right)}{\text{Ln } 0.995} = \frac{12 \text{ Ln } 0.5}{\text{Ln } 0.995}. \qquad (a)$$

To find the value of Ln 0.5,
push hairline to 0.5 on *Ln-2*,
under hairline read 693 on D.

In accord with the legend of the *Ln-2* scale, Ln 0.5 = −0.693.

To find the value of Ln 0.995,
push hairline to 0.995 on *Ln-0*,
under hairline read 502 on D.

In accord with the legend of the *Ln-0* scale, Ln 0.995 = −0.00502.

Substitute the above values for Ln 0.5 and Ln 0.995 in (a) to obtain

$$t = \frac{(12)\ (-0.693)}{-0.00502}.$$

*The half life of a radioactive substance is defined as the time it takes for 50% of the substance to disappear.

To find t,

push hairline to 12 on D,
draw 502 of C under the hairline,
push hairline to 693 on C,
under hairline read 1657 on D.
Therefore, $t = 1657$ years.

The position of the decimal point in the answer was determined by a mental estimate.

EXERCISES

1. Find the Ln of: 500, 50, 2, 1.4, 1.043.

2. Find the Ln of: 0.002, 0.02, 0.5, 0.714, 0.9091, 0.9804.

3. Find the values of:

(*a*) Ln 76.	(*d*) Ln 0.84.	(*g*) Ln 0.909.
(*b*) Ln 7.6.	(*e*) Ln 0.145.	(*h*) Ln 1.43.
(*c*) Ln 9.2.	(*f*) Ln 0.893.	(*i*) Ln 1.043.

4. Show that the first three significant digits in the Ln of each of the following numbers is 693: 1.0718, 0.9330, 2, 0.5, 1024, 0.000977, 0.99307.

5. Find Ln 4. Then find seven numbers other than 4, each having as its Ln the same first three figures as Ln 4.

6. Find the half life of a radioactive substance if 25% of it disappears in 10 years. Use the formula in Example 2.

7. If 30% of a radioactive substance disappears in 10 days, how long will it take for 90% to disappear? Use the formula in Example 2.

8. Bacteria in a certain culture increase at a rate proportional to the number present. If the original number increases 50% in $\frac{1}{2}$ hour, in how many hours can one expect three times the original number? Use the formula in Example 2.

9. The time (t) required, in years, for a sum of money to double itself if interest is compounded continuously at r per cent per annum is

$$t = \frac{100 \text{ Ln } 2}{r}.$$

How long will it take if the rate of interest is 6 per cent per annum?

10. The neper (α) is a unit of measure used in communication systems. It is defined by

$$\alpha = \text{Ln}\frac{N_1}{N_2} \qquad (b)$$

where $\frac{N_1}{N_2}$ represents a ratio between two acoustic or electric quantities of the same kind measured in the same units.

Using formula (*b*), find the number of nepers corresponding to a voltage ratio of 2.10.

65. Powers of e. We have seen that we can locate a number on a Lon scale and opposite it, on D, read its natural logarithm. By the inverse process, we can locate a number x on the D scale and opposite it read e^x on a Lon scale.

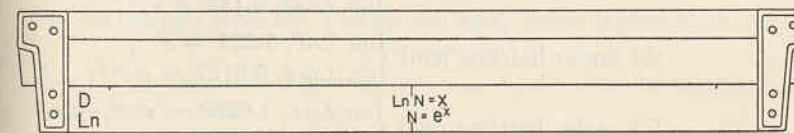


FIG. 67.

The reason for this relationship is illustrated in Fig. 67. Since Ln $N = x$, then by definition $N = e^x$. But from §64, Ln N on D is opposite N on a Lon scale. Therefore, x on D is opposite e^x on a Lon scale. The following rule embodies this relation:

Rule. *To find the value of e^x, push hairline to x on D and under the hairline, read the value of e^x on the Lon scale which contains x between its legend numbers.*

Example 1. Evaluate $e^{3.5}$ and $e^{-3.5}$.

Solution. In accord with the above rule (see Fig. 68) make the following setting:

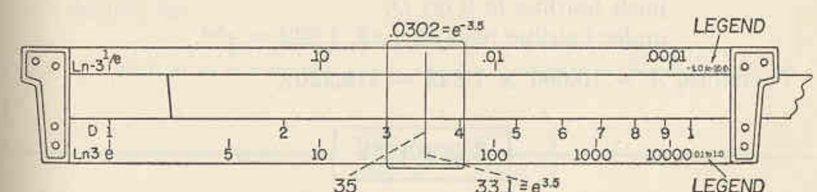


FIG. 68.

push hairline to 35 on D,
under hairline read on *Ln3*, $33.1 = e^{3.5}$.
Reverse the rule,
under the hairline read on *Ln-3*, $0.0302 = e^{-3.5}$.

Scale *Ln3* was selected on which to read the value of $e^{3.5}$ because the exponent 3.5 lies within the range specified by the legend of the *Ln3* scale. Similarly, scale *Ln-3* was chosen for $e^{-3.5}$ because -3.5 lies within the range specified by the legend on the *Ln-3* scale.

Example 2. Evaluate: (*a*) e^2, e^{-2}; (*b*) $e^{0.2}$, $e^{-0.2}$; (*c*) $e^{0.02}$, $e^{-0.02}$; (*d*) $e^{0.002}$, $e^{-0.002}$.

Solution: Push hairline to 2 on *D*,

(*a*) under hairline read $\begin{cases}\text{on } Ln3,\ 7.39 = e^2, \\ \text{on } Ln\text{-}3,\ 0.135 = e^{-2};\end{cases}$

(*b*) under hairline read $\begin{cases}\text{on } Ln2,\ 1.221 = e^{0.2}, \\ \text{on } Ln\text{-}2,\ 0.8187 = e^{-0.2};\end{cases}$

(*c*) under hairline read $\begin{cases}\text{on } Ln1,\ 1.0202 = e^{0.02}, \\ \text{on } Ln\text{-}1,\ 0.9802 = e^{-0.02};\end{cases}$

(*d*) under hairline read $\begin{cases}\text{on } Ln0,\ 1.002 = e^{0.002}, \\ \text{on } Ln\text{-}0,\ 0.998 = e^{-0.002}.\end{cases}$

Example 3. The amount *A* of money accumulated by investing A_0 dollars at *r* per cent compounded continuously is, at the end of *t* years,

$$A = A_0 e^{\frac{r}{100}t}.$$

Find the amount accumulated in ten years when $10,000 is invested at 6% interest compounded continuously.

Solution: The above formula adapted to the given quantities becomes $A = 10{,}000\ e^{.06(10)} = 10{,}000\ e^{0.6}$. Hence, push hairline to 6 on *D*,
under hairline read on *Ln2*, $1.822 = e^{0.6}$.

Therefore, $A = 10{,}000 \times 1.822 = \$18{,}220$.

EXERCISES

1. Evaluate:
(*a*) e^3. (*b*) e^{-3}. (*c*) $e^{0.4}$. (*d*) $e^{-0.4}$.
(*e*) $e^{0.035}$. (*f*) $e^{-0.035}$. (*g*) $e^{1.342}$. (*h*) $e^{-1.342}$.
(*i*) $e^{-2.46}$. (*j*) $e^{3.55}$. (*k*) $e^{-0.0264}$. (*l*) $e^{0.0853}$.

2. Find e^x when: (*a*) $x = 2.12$; (*b*) $x = -2.12$; (*c*) $x = 0.212$; (*d*) $x = -0.212$; (*e*) $x = 0.0212$; (*f*) $x = -0.0212$.

3. Evaluate:
(*a*) e^4. (*b*) $e^{8.2}$. (*c*) $e^{0.43}$.
(*d*) $e^{0.0214}$. (*e*) $e^{-3.4}$. (*f*) $e^{-0.163}$.
(*g*) $e^{-0.0185}$. (*h*) $e^{-6.2}$.

4. Evaluate:
(*a*) $e^{\sin 45°}$. (*b*) $e^{-\sin 45°}$. (*c*) $e^{-\cos 65°}$. (*d*) $e^{\cos 65°}$. (*e*) $e^{\tan 40°}$.
(*f*) $e^{-\tan 40°}$. (*g*) $e^{\sqrt{5}}$. (*h*) $e^{-\sqrt{5}}$. (*i*) $e^{-\sqrt{0.142}}$. (*j*) $e^{\sqrt{0.142}}$.
(*k*) $e^{\cot 60°}$. (*l*) $e^{\cot 120°}$. (*m*) $e^{\sqrt[3]{33.3}}$. (*n*) $21e^{\tan 12°}$. (*o*) $e^{1.8 \sin 26°}$.

5. The damping factor *f* for a certain oscillatory motion is given by the formula

$$f = e^{-0.04t},$$

where *t* is the time in seconds. Find the time elapsed while the damping factor changes from 1 to ½.

6. A tank is filled with 10 gallons of brine in which is dissolved 5 lb. of salt. Brine containing 3 lb. of salt per gallon enters the tank at 2 gallons per minute and the well-stirred mixture leaves the tank at the same rate. The amount *A* of salt in the tank at any time *t* minutes is

$$A = 30 - 25\ e^{-0.2t}\ \text{lb.}$$

Find how much salt is present in the tank after 22 minutes.

---

66. Hyperbolic functions. Certain combinations of exponential functions, having relations to each other analogous to the relations among the trigonometric functions, are called hyperbolic functions. The three hyperbolic functions most frequently used are called **hyperbolic sine of x, hyperbolic cosine of x,** and **hyperbolic tangent of x.** They are designated by *sinh x*, *cosh x* and *tanh x*, respectively, and are defined by

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{2x} - 1}{e^{2x} + 1}$$

The hyperbolic functions are used in many fields and especially in discussions relating to the catenary, a curve in which the cables of suspension bridges hang.

Example. Evaluate sinh 2.14.

Solution: From the above definition $\sinh 2.14 = \dfrac{e^{2.14} - e^{-2.14}}{2}$

To find the values of $e^{2.14}$ and $e^{-2.14}$,
push hairline to 214 on D
under hairline read $\begin{cases} \text{on } Ln3,\ 8.50 = e^{2.14}, \\ \text{on } Ln\text{-}3,\ 0.118 = e^{-2.14}. \end{cases}$

$$\text{Therefore, } \sinh 2.14 = \frac{8.50 - 0.118}{2} = \mathbf{4.191.}$$

EXERCISES

Using the above definitions evaluate the following hyperbolic functions:

1. sinh 1.04.
2. cosh 2.52.
3. tanh 1.41.
4. sinh 0.75.
5. cosh 0.55.
6. tanh 0.61.

7. The formulas for the length l and the sag s (see Fig. 69) of a uniform chain hung from two points on the same level are

$$l = 2\frac{H}{w}\sinh\frac{wb}{H}, \qquad s = \frac{H}{w}\left(\cosh\frac{wb}{H} - 1\right)$$

where w is wt. per ft. of chain, H is the tension at the lowest point, and $2b$ is the distance between the points of suspension, called the span. Find the length l and the sag s if $w = 2$ lb./ft., $H = 26$ lb., and $b = 30$ ft.

8. A cable weighing 4 lb. per ft. has a span of 550 ft. The tension at the lowest point is 1600 lb. Find the sag s and the length l of the cable.

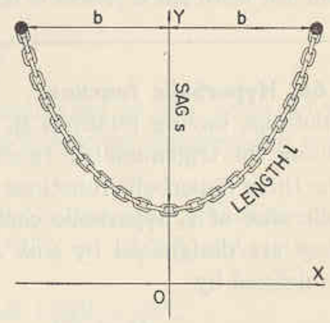


FIG. 69

67. Powers-of-ten notation. Problems of science often involve very large numbers and very small numbers. For ease in reading or combining such numbers, they are often written in the form

$$m \times 10^k$$

where m is an ordinary number, usually a number between 1 and 10, and k is a positive or negative integer. Examples of numbers in this form are:

$$384 = 3.84 \times 10^2, \qquad 384000000 = 3.84 \times 10^8,$$
$$0.0384 = 3.84 \times 10^{-2}, \qquad 0.0000384 = 3.84 \times 10^{-5}.$$

Observe that *in powers-of-ten notation the factor 10^k has the effect of moving the decimal point k places to the right when k is positive, or k places to the left when k is negative.*

In combining numbers in the powers-of-ten notation, the following laws of exponents apply (see §61):

$$10^k \times 10^r = 10^{k+r}, \quad \frac{10^k}{10^r} = 10^{k-r}, \quad (10^k)^r = 10^{kr}.$$

For example,

$$\frac{832000 \times 0.00324}{5230} = \frac{8.32 \times 10^5 \times 3.24 \times 10^{-3}}{5.23 \times 10^3}$$
$$= \frac{8.32 \times 3.24}{5.23} \times 10^{5+(-3)+(-3)} = 5.15 \times 10^{-1} = \mathbf{0.515.}$$

$$\frac{(160000)^5 \times (0.00000195)^6}{(0.00545)^2} = \frac{(1.6 \times 10^5)^5 \times (1.95 \times 10^{-6})^6}{(5.45 \times 10^{-3})^2}$$
$$= \frac{(1.6)^5\,(1.95)^6}{(5.45)^2} \times 10^{5(5)-6(-6)-(2)(-3)} = \frac{10.49 \times 55.0 \times 10^{-5}}{29.7}$$
$$= 19.4 \times 10^{-5} = 1.94 \times 10^{-4} = \mathbf{0.000194.}$$

EXERCISES

1. Express in the powers-of-ten notation:
   (*a*) 5860. (*d*) 0.479. (*g*) 0.00000091.
   (*b*) 675000. (*e*) 28 million. (*h*) 0.00495×10^6.
   (*c*) 0.0623. (*f*) 2.76 billion. (*i*) 8645×10^{-6}.

2. Simplify, and give the answer in the powers-of-ten notation:
   (*a*) $(3 \times 10^2)(6 \times 10^4)$. (*d*) $(5 \times 10^{-6})(7 \times 10^4)^2$.
   (*b*) $(7 \times 10^{-3})^3$. (*e*) $\sqrt{64 \times 10^8}$.
   (*c*) $(3 \times 10^{-2})^3$. (*f*) $\sqrt[3]{8 \times 10^{-12}}$.

3. To make each of the following evaluations, express the numbers in the powers-of-ten notation, simplify, make slide rule computations, and finally write the answer in the powers-of-ten notation:
   (*a*) $(8.31 \times 10^{-4})^2$. (*d*) $(3.16 \times 10^4)^{0.75}$.
   (*b*) $(7.45 \times 10^5)^2\sqrt{2.65 \times 10^{-4}}$. (*e*) $\dfrac{(2.30 \times 10^2)^3\,(1.42 \times 10^{-3})^4}{(1.96 \times 10^{-3})^2}$.
   (*c*) $\dfrac{\sqrt{3.68 \times 10^6}}{\sqrt{5.12 \times 10^{-4}}}$.

4. Evaluate and give the answer in the powers-of-ten notation:

(*a*) $\sqrt[3]{64 \times 10^3 \times 0.125 \times 10^{-5}}$. (*d*) $\sqrt[4]{25600 \times 0.00816}$.

(*b*) $6.8 \times 10^4 \sqrt{2.96 \times 10^{-4}}$. (*e*) $\dfrac{(0.000659)^{1/4}}{0.00673}$.

(*c*) $\dfrac{\sqrt{72.4 \times 10^2}}{\sqrt{0.00042 \times 0.0683}}$. (*f*) $\dfrac{(2675000)^{5/16}}{(0.00234)^{1/5}}$.

5. A light year is the distance traveled by light in a year. If the speed of light is 186,300 mi./sec., find in the powers-of-ten notation the number of miles in a light year.

68. Mated scales; reciprocals. Two Lon scales having the same number in their labels are called *mated scales,* and each of the pair is the *mate* of the other. The mated pairs are:

Ln0	*Ln1*	*Ln2*	*Ln3*
Ln-0	*Ln-1*	*Ln-2*	*Ln-3*

Rule. *Opposite numbers on mated scales are reciprocals of each other.*

Thus to find the reciprocal of 1.00203:
push hairline to 1.00203 on *Ln0*,
under hairline read **0.997975** on *Ln-0*.

To find the reciprocal of 0.552:
push hairline to 0.552 on *Ln-2*,
under hairline read **1.812** on *Ln2*.

Note that in finding reciprocals by means of mated Lon scales, there is no question as to the position of the decimal point.

EXERCISES

1. Set the hairline to 5 on *Ln3*, read the reciprocal of 5 on *Ln-3*; set the hairline to 1.25 on *Ln2*, read the reciprocal of 1.25 on *Ln-2*; set the hairline to 1.04 on *Ln1*, read the reciprocal of 1.04 on *Ln-1*.

2. Using the Lon scales find the reciprocal of each of the following:

(*a*) 16.	(*d*) 1.95.	(*g*) 1.0142.
(*b*) 3.52.	(*e*) 0.752.	(*h*) 0.9515.
(*c*) 0.0155.	(*f*) 1.163.	(*i*) 1.0075.

69. Powers of any number: elementary. In §65 the method of finding powers of *e* was considered. Powers of any other number can

be found by a similar method. To illustrate the process, consider the problem of finding the value of 3^2 and 3^{-2} (see Fig. 70).

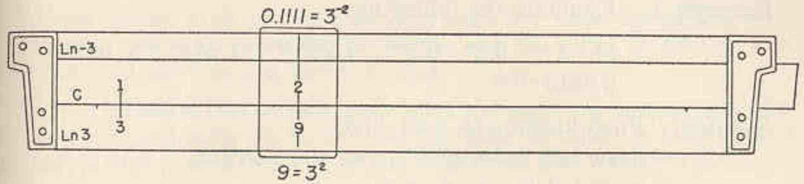


FIG. 70.

Push hairline to 3 on *Ln3*,
draw left index of *C* under the hairline,
push hairline to 2 on *C*,
under hairline read $\begin{cases}\text{on } Ln3,\ 9 = 3^2, \\ \text{on } Ln\text{-}3,\ 0.1111 = 3^{-2}.\end{cases}$

Without changing the position of the slide, push the hairline to 3, 5, and 7 on *C* and under the hairline read, on *Ln3*, $27 = 3^3$, $243 = 3^5$, and $2187 = 3^7$.

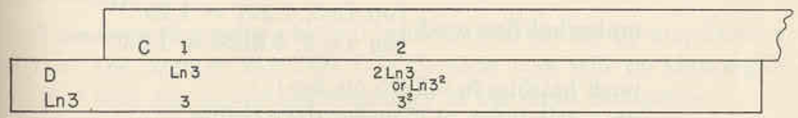


FIG. 71.

The above setting can be explained as follows (see Fig. 71): by pushing the hairline to 3 on *Ln3*, the $\text{Log}_e 3$ is found on *D* under the hairline. Next by setting the left index of *C* under the hairline and pushing the hairline to 2 on *C*, the $\text{Log}_e 3$ is multiplied by 2 to obtain $2 \text{ Log}_e 3$ under the hairline on *D*. $2 \text{ Log}_e 3$ can be written in the form $\text{Log}_e 3^2$ or $\text{Ln } 3^2$. Hence 3^2 is found on *Ln3* under the hairline, since by §64 $\text{Ln } 3^2$ on *D* is opposite 3^2 on *Ln3*.

The following rule covers the process just illustrated:

Rule. *To find the power d^n, push hairline to d on a Lon scale, draw index of C under the hairline, push hairline to n on C, under hairline read d^n on a Lon scale.*

In the following examples, the proper scale on which each answer is to be read is specified. The process for determining the proper scale for any similar problem will be discussed in the next article.

Example 1. Evaluate the following:

3^4, 3^{-4}, 0.2^5, 0.2^{-5}, $1.25^{0.932}$, $1.25^{-0.932}$, $0.9615^{12.5}$, $0.9615^{-12.5}$.

Solution: Push hairline to 3 on *Ln3*,
draw left index of *C* under the hairline,
push hairline to 4 on *C*,

under hairline read $\begin{cases}\text{on } Ln3,\ 81 = 3^4,\\ \text{on } Ln\text{-}3,\ 0.0123 = 3^{-4};\end{cases}$

push hairline to 0.2 on *Ln-3*,
draw left index of *C* under the hairline,
push hairline to 5 on *C*,

under hairline read $\begin{cases}\text{on } Ln\text{-}3,\ 0.00032 = 0.2^5\\ \text{on } Ln3,\ 3100 = 0.2^{-5};\end{cases}$

push hairline to 1.25 on *Ln2*,
draw right index of *C* under the hairline,
push hairline to 932 on *C*,

under hairline read $\begin{cases}\text{on } Ln2,\ 1.231 = 1.25^{0.932}\\ \text{on } Ln\text{-}2,\ 0.8124 = 1.25^{-0.932};\end{cases}$

push hairline to 0.9615 on *Ln-1*,
draw left index of *C* under the hairline,
push hairline to 125 on *C*,

under hairline read $\begin{cases}\text{on } Ln\text{-}2,\ 0.612 = (0.9615)^{12.5}\\ \text{on } Ln2,\ 1.633 = (0.9615)^{-12.5}.\end{cases}$

EXERCISES

1. Using the settings indicated by the italicized rule in §69, find on *Ln3* the values of 2^2, 5^2, 7^2, 8^2, and on *Ln-3* the values of 2^{-2}, 5^{-2}, 7^{-2}, 8^{-2}.

2. Using the settings indicated in the italicized rule, find on *Ln-2* the values of $0.88^{2.1}$, $0.80^{2.1}$, $0.65^{2.1}$, and on *Ln2* the values of $0.88^{-2.1}$, $0.80^{-2.1}$, $0.65^{-2.1}$.

3. Push hairline to 3 on *Ln3* and under the hairline read on *Ln-3*, 3^{-1}; on *Ln2*, $3^{0.1}$; on *Ln-2*, $3^{-0.1}$; on *Ln1*, $3^{0.01}$; on *Ln-1*, $3^{-0.01}$.

4. Use the rule of tbis article to find 3^2, 3^{-2}, 4^2, 4^{-2}, 6^2, 6^{-2}, 9^2, 9^{-2}, 5.5^3, 5.5^{-3}.

5. Push hairline to 1.06 on *Ln1* and under hairline read the values of 1.06^{-1} on *Ln-1*, 1.06^{10} on *Ln2*, 1.06^{-10} on *Ln-2*, 1.06^{100} on *Ln3*, and 1.06^{-100} on *Ln-3*.

6. Push hairline to 0.1 on *Ln-3* and under hairline read the values of $(0.1)^{-1}$ on *Ln3*, $(0.1)^{0.1}$ on *Ln-2*, $(0.1)^{-0.1}$ on *Ln2*, $(0.1)^{0.01}$ on *Ln-1*, $(0.1)^{-0.01}$ on *Ln1*.

7. Push hairline to 25 on *Ln3*, draw right index of *C* under the hairline, push the hairline to 5 on *C*, and under the hairline read the values of: $25^{0.5}$ on *Ln3*, $25^{-0.5}$ on *Ln-3*; $25^{0.05}$ on *Ln2*, $25^{-0.05}$ on *Ln-2*.

8. Push hairline to 0.84 on *Ln-2*, draw index of *C* under the hairline, push hairline to 3 on *C*, and under the hairline read the values of: $(0.84)^3$ on *Ln-2*, $(0.84)^{-3}$ on *Ln2*, $(0.84)^{30}$ on *Ln-3* $(0.84)^{-30}$, on *Ln3*, $(0.84)^{0.3}$ on *Ln-1*, $(0.84)^{-0.3}$ on *Ln1*.

---

70. Powers of any number: general. In finding a power of any number — such as $1.2^{14.5}$ or $12^{-0.0024}$ — it is often necessary to locate the base on one Lon scale and read the power on another Lon scale. In the problems in the preceding article, the scale on which to read the power was specified. In this article we shall discuss the complete process for determining which scale to use when finding any power of any number, provided that no step in the process involves any number outside the range of the Lon scales. These scales represent numbers from 0.00003 to 30,000 except for a gap between 0.999 and 1.001. Calculations involving numbers not on the Lon scales are discussed in §§76 and 77.

These operating principles will be developed in step-by-step fashion, so that the user of the DECI-LON slide rule will be thoroughly familiar with the power and flexibility of the eight Lon scales.

A. The "Same Scale" Principle: Consider the process of finding 1.2^2. The base number 1.2 is located on the *Ln2* scale. Hence, we set the left index of *C* opposite 1.2 on *Ln2*. Thus

1.2^1 on *Ln2* is opposite 1 on *C*.

Now push the hairline to 2 on *C* and under the hairline read
$1.44 = 1.2^2$ on *Ln2*.

Hence, 1.2^2 on *Ln2* is opposite 200 on *C*.

Note that scale *Ln2* is the same scale on which the base 1.2 was found. Similarly,

opposite 300 on *C* read $1.73 = 1.2^3$ on *Ln2*,
opposite 400 on *C* read $2.07 = 1.2^4$ on *Ln2*,
opposite 504 on *C* read $2.50 = 1.2^{5.04}$ on *Ln2*.

In each of the above settings, note the close association between the number on C and the exponent of 1.2. Also note that all these readings are on *Ln2*, *the same scale on which the base* 1.2 *was found.*

Now interchange the indexes of the C scale; push the hairline to 700 on C and

opposite 700 on C read $1.136 = 1.2^{0.7}$ on *Ln2*.

Similarly,

opposite 600 on C read $1.1156 = 1.2^{0.6}$ on *Ln2*.

Again observe the close association between the number read on C and the exponent of 1.2; note that the readings are still made on *Ln2*, on which 1.2 appears.

These examples illustrate the operating principle (Fig. 72):

Fig. 72.

The "Same Scale" Principle: To find a power of a base number, N, locate the base number on a Lon scale and set an index of C opposite it. Then any number to the right of N on the same Lon scale on which N appears is a power of N having an exponent whose value lies between 1 and 10 and whose digits are the same as the digits of the number on the C scale opposite this power. Moreover, any number to the left of N on the same Lon scale on which N appears is a power of N having an exponent whose value lies between 0.1 and 1 and whose digits are the same as the digits of the number on the C scale opposite this power.

EXERCISES

1. Evaluate $6^{2.12}$, $6^{3.68}$, $6^{5.52}$.
2. Evaluate $6^{0.915}$, $6^{0.714}$, $6^{0.586}$.
3. Evaluate $1.322^{2.46}$, $1.322^{3.45}$, $1.322^{0.362}$, $1.322^{0.552}$, $1.322^{0.892}$, $1.322^{0.916}$.
4. Evaluate $1.022^{1.53}$, $1.022^{3.450}$, $1.022^{4.21}$, $1.022^{0.920}$, $1.022^{0.746}$, $1.022^{0.482}$.
5. Evaluate $1.0075^{1.31}$, $1.0075^{0.922}$, $1.0075^{0.634}$, $1.0075^{0.412}$, $1.0075^{0.212}$, $1.0075^{0.1422}$.

B. The "Tenth Power" Principle: In considering the second operating principle, keep in mind the distinction between the "label" of a scale and the "legend" of a scale. The label, or name, of the *Ln3* scale is "Ln3"; its legend is "1 to 10."

As we saw in §64, the legend of each Lon scale gives the range of the exponents of the powers of e found on that scale. For example,

Scale *Ln1* consists of numbers from $e^{0.01}$ to $e^{0.1}$
" *Ln2* " " " " $e^{0.1}$ " e^{1}
" *Ln3* " " " " e^{1} " e^{10}.

Hence, any number on *Ln3* is the 10th power of any number opposite it on *Ln2*; any number on *Ln2* is the 10th power of its opposite number on *Ln1*.

To illustrate, again set the left index of C opposite 1.2 on *Ln2*. Push the hairline to 200 on C. By the "Same Scale" principle, the number under the hairline on *Ln2* is 1.2^2.

For the reason just discussed, the number

$$\text{under the hairline } \begin{cases} \text{on } Ln3 \text{ is } 1.2^{20}, \\ \text{on } Ln1 \text{ is } 1.2^{0.2}, \\ \text{on } Ln0 \text{ is } 1.2^{0.02}. \end{cases}$$

Note that in moving along the hairline from one Lon scale to an adjacent Lon scale with the higher number in its label, the exponent of the power of 1.2 is multiplied by 10; in moving to the adjacent Lon scale with the lower number in its label, the exponent is divided by 10.

This illustrates the second operating principle (Fig. 73):

Ln-3	N^{-1} to N^{-10}		N^{-10} to N^{-100}
Ln-2	$N^{-0.1}$ to N^{-1}		N^{-1} to N^{-10}
Ln-1	$N^{-0.01}$ to $N^{-0.1}$		$N^{-0.1}$ to N^{-1}
Ln-0	$N^{-0.001}$ to $N^{-0.01}$		$N^{-0.01}$ to $N^{-0.1}$
C	6 7 8 9 1		2 3
Ln 0	$N^{0.001}$ to $N^{0.01}$		$N^{0.01}$ to $N^{0.1}$
Ln 1	$N^{0.01}$ to $N^{0.1}$		$N^{0.1}$ to N^{1}
Ln 2	$N^{0.1}$ to N^{1}	N	N^{1} to N^{10}
Ln 3	N^{1} to N^{10}		N^{10} to N^{100}

Fig. 73.

The "Tenth Power" Principle: Each number on any Lon scale is the 10th power of the number opposite it on the adjacent Lon scale with the lower number in its label.

EXERCISES

1. Evaluate $6^{2.12}$, $6^{0.212}$, $6^{0.0212}$, $6^{0.00212}$.
2. Evaluate $1.0075^{0.510}$, $1.0075^{5.10}$, $1.0075^{51.0}$.
3. Evaluate $1.025^{4.02}$, $1.025^{0.402}$, $1.025^{40.2}$.
4. Evaluate $1.45^{2.52}$, $1.45^{0.252}$, $1.45^{0.0252}$, $1.45^{25.2}$.
5. Evaluate $1.25^{0.146}$, $1.25^{40.3}$, $1.25^{9.44}$.

***C. The "Mated Scale" Principle*:** In §68 it was shown that opposite numbers on mated Lon scales are reciprocals of each other. Since N^P and N^{-P} are reciprocals, this relationship (Fig. 74) can be restated as:

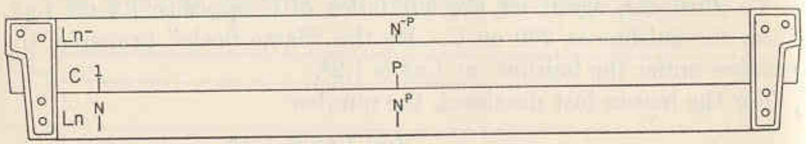


FIG. 74.

The "Mated Scale" Principle: When the hairline is set to N^P on any Lon scale, it is also set to N^{-P} on the mated Lon scale.

As an illustration of this principle set the left index of C opposite 1.2 on *Ln2* and opposite 2 on C read the powers in the left-hand column of the table below in accord with the "Same Scale" and "Tenth Power" principles, and those in the right-hand column in accord with the "Mated Scale" principle:

$1.2^{0.02}$ on *Ln0*	$1.2^{-0.02}$ on *Ln-0*
$1.2^{0.2}$ on *Ln1*	$1.2^{-0.2}$ on *Ln-1*
1.2^{2} on *Ln2*	1.2^{-2} on *Ln-2*
1.2^{20} on *Ln3*	1.2^{-20} on *Ln-3*

EXERCISES

1. Evaluate $3.75^{4.21}$, $3.75^{-4.21}$, $3.75^{6.22}$, $3.75^{-6.22}$, $3.75^{-5.04}$.
2. Evaluate $1.024^{3.62}$, $1.024^{0.362}$, $1.024^{-0.362}$, $1.024^{-15.6}$.
3. Evaluate $1.0075^{-4.22}$, $1.0075^{-3.42}$, $1.0075^{-50.6}$.
4. Evaluate $0.842^{-2.31}$, $0.842^{-18.1}$, $0.842^{3.42}$.
5. Evaluate $0.9982^{-15.4}$, 0.9982^{-165}, $0.9982^{-2.81}$.

***D. Index Power Ranges*:** The relations set forth in the "Same Scale," "Tenth Power" and "Mated Scale" principles are sufficient to determine the proper Lon scale on which to read a power of N. One further device will be presented here which on certain occasions will be found exceedingly helpful. This device makes use of the interrelationship of the indexes of the Lon scales, by means of which the ranges for the powers of any given base number N are established on the various Lon scales.

Again set the left index of C opposite 1.2 on *Ln2* and read 548 on C opposite the right index of D. By the "Same Scale" principle the number at the right index of *Ln2* is $1.2^{5.48}$. Hence the numbers on *Ln2* to the right of 1.2 are powers of 1.2 having exponents which range from 1 to 5.48.

Interchange the indexes of the C scale and find opposite the left index of *Ln2* the same digits 5-4-8 on C. By the "Same Scale" principle the number at the left index of *Ln2* is $1.2^{0.548}$ and the numbers on *Ln2* to the left of 1.2 are powers of 1.2 which range from 0.548 to 1. It follows, therefore, that the powers of 1.2 which can be read on scale *Ln2* have exponents which range from **0.548** to **5.48**.

By the "Tenth Power" and "Mated Scale" principles, it can be established that the powers of 1.2 which can be read on *Ln3* have exponents which range from 5.48 to 54.8, and the powers of 1.2 that can be read on *Ln-3* have exponents which range from -5.48 to -54.8. In like manner, the ranges for the other Lon scales can be determined. These ranges will be referred to as the *"Index Power Ranges."*

When it is necessary to find various powers of the same base number, it is often convenient to list the "Index Power Ranges" of the Lon scales and use them like scale legends. For example, to write a set of "Index Power Ranges" for determining the proper scale on which to read various powers of 9, set the index of C opposite 9 on *Ln3*. Opposite the index of D we find, on C, the digits 4, 5, 5. By applying the three operating principles just discussed, we write these "Index Power Ranges" for powers of 9:

	Left index		Right index
Ln-3	$9^{-0.455}$	. . . to . . .	$9^{-4.55}$
Ln-2	$9^{-0.0455}$	. . . to . . .	$9^{-0.455}$
Ln-1	$9^{-0.00455}$	. . . to . . .	$9^{-0.0455}$
Ln-0	$9^{-0.000455}$	. . . to . . .	$9^{-0.00455}$
Ln0	$9^{0.000455}$	. . . to . . .	$9^{0.00455}$
Ln1	$9^{0.00455}$	. . . to . . .	$9^{0.0455}$
Ln2	$0^{0.0455}$	. . . to . . .	$9^{0.455}$
Ln3	$9^{0.455}$	. . . to . . .	$9^{4.55}$

In each case, of course, the actual digits in the exponents of the powers of 9 are located on the *C* scale.

Example 1. Find the values of the powers of 9 whose exponents are 0.545, 2.13, −2.13, 0.213, −0.213, 0.0213, −0.0213, 0.00213, −0.00213.

Solution: In this example, let us determine the scale on which to read each answer by referring to the "Index Power Ranges" above for the powers of 9. Thus we choose *Ln3* as the scale on which to read the values of $9^{0.545}$ and $9^{2.13}$ since these powers of 9 lie within the "Index Power Range" $9^{0.455}$ to $9^{4.55}$ of the *Ln3* scale (see Fig. 75).

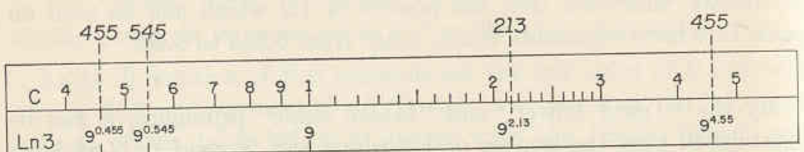


Fig. 75.

Also we read the value of $9^{-2.13}$ on scale *Ln-3*, the mate of *Ln3*. Similarly we choose *Ln2* and its mate *Ln-2* as the scales on which to read the values of $9^{0.213}$ and $9^{-0.213}$, respectively. In a like manner, we use the "Index Power Ranges" to select the proper scale on which to read the values of the remaining powers of 9. We now make the following setting:

push hairline to 9 on *Ln3*,
draw right index of *C* under the hairline,
push hairline to 545 on *C*,
under hairline read on *Ln3*, $3.31 = 9^{0.545}$;
interchange indexes of *C*,
under hairline,

read $\begin{cases}\text{on } Ln3,\ 108 = 9^{2.13}, \\ \text{on } Ln\text{-}3,\ 0.0092 = 9^{-2.13};\end{cases}$

read $\begin{cases}\text{on } Ln2,\ 1.597 = 9^{0.213}, \\ \text{on } Ln\text{-}2,\ 0.626 = 9^{-0.213};\end{cases}$

read $\begin{cases}\text{on } Ln1,\ 1.0479 = 9^{0.0213}, \\ \text{on } Ln\text{-}1,\ 0.9543 = 9^{-0.0213};\end{cases}$

read $\begin{cases}\text{on } Ln0,\ 1.0047 = 9^{0.00213}, \\ \text{on } Ln\text{-}0,\ 0.99532 = 9^{-0.00213}.\end{cases}$

It is to be emphasized that the last eight exponents in the example have the same significant digits and that all of these answers appear opposite each other at the same hairline setting (see Fig. 76).

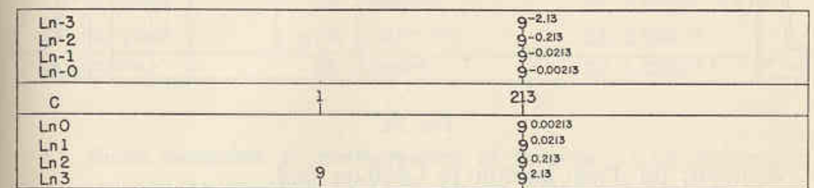


Fig. 76.

Moreover, each number on the higher labeled scale is the 10th power of the number opposite it on the lower labeled adjacent scale. Finally, observe that exponents having the same digits but opposite in sign are on mated scales.

Example 2. Evaluate

(*a*) $(5.27)^{0.044}$, (*b*) $(0.955)^{186.3}$, (*c*) $(1.456)^{-0.054}$.

Solution: (*a*) Push hairline to 5.27 on *Ln3*,
draw left index of *C* under the hairline,
push hairline to 440 on *C*,
under hairline read on *Ln1*, $1.0759 = (5.27)^{0.044}$.

Note that, after the setting was made, in accordance with the "Same Scale" principle, $5.27^{4.4}$ was on *Ln3* at the hairline, and that by the "Tenth Power" principle $5.27^{0.44}$ was on *Ln2*, while $5.27^{0.044}$ was on *Ln1*. Fig. 77 shows the setting.

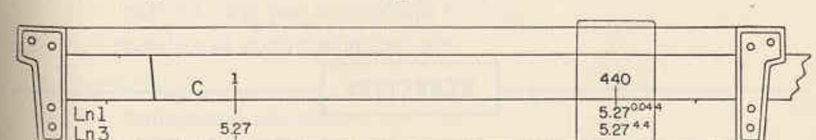


Fig. 77.

Solution: (*b*) Push hairline to 0.955 on *Ln-1*,
draw left index of *C* under the hairline,
push hairline to 1863 on *C*,
under hairline read on *Ln-3*, **0.000188** = $(0.955)^{186.3}$.

Note that, after the setting was made, in accordance with the "Same Scale," principle $0.955^{1.863}$ was on *Ln-1* at the hairline, and that by the "Tenth Power" principle, $0.955^{18.63}$ was on *Ln-2* and $0.955^{186.3}$ was on *Ln-3*. Fig. 78 shows the setting.

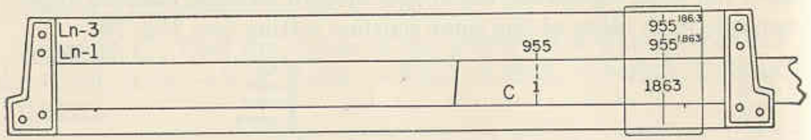


FIG. 78.

Solution: (*c*) Push hairline to 1.456 on *Ln2*,
draw right index of *C* under the hairline,
push hairline to 540 on *C*,
under hairline read on *Ln-1*, **0.97995** = $(1.456)^{-0.054}$.

Note that, after the setting was made, $1.456^{0.54}$ was at the hairline on *Ln2* by the "Same Scale" principle; $1.456^{0.054}$ was at the hairline on *Ln1* by the "Tenth Power" principle, and $1.456^{-0.054}$ was under the hairline on *Ln-1* by the "Mated Scale" principle. Fig. 79 shows the setting.

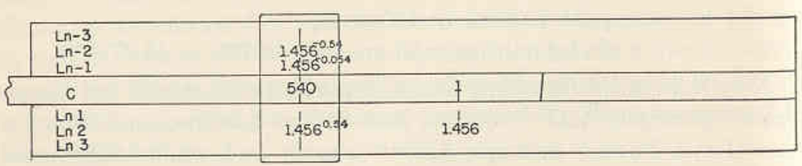


FIG. 79.

EXERCISES

1. Find the values of: $(1.056)^{0.85}$, $(1.056)^{8.5}$, $(1.056)^{-85}$.

2. Find the values of:

(*a*) $1.03^{1.81}$.	$1.03^{-1.81}$.	$1.03^{-18.1}$.
(*b*) $8.55^{1.81}$.	$8.55^{-1.81}$.	$8.55^{0.0181}$.
(*c*) $0.77^{2.11}$.	$0.77^{21.1}$.	$0.77^{-0.211}$.
(*d*) $0.224^{0.843}$.	$0.224^{-0.00843}$.	$0.224^{-0.843}$.

Evaluate the following expressions:

3. $1.03^{18.1}$.	12. $74^{0.004}$.	21. $9^{\sin 30°}$.
4. $1.0163^{57.5}$.	13. $74^{-0.04}$.	22. $9^{-\sin 30°}$.
5. $8.55^{0.18}$.	14. 0.74^{10}.	23. $5^{\cos 28°}$.
6. $0.98^{1.783}$.	15. $0.74^{-2.88}$.	24. $5^{-\cos 28°}$.
7. $0.98^{178.3}$.	16. $0.74^{0.657}$.	25. $1.6\sqrt[3]{5}$.
8. $0.98^{-178.3}$.	17. $1.035^{-1.685}$.	26. $0.675\sqrt[3]{0.00465}$.
9. $2.72^{+2.43}$.	18. $1.035^{168.5}$.	27. 0.983^{150}.
10. $2.72^{-2.43}$.	19. $1.035^{-0.865}$.	28. 0.983^{-15}.
11. $74^{0.04}$.	20. 0.988^{243}.	29. $1.0325^{42.5}$.

71. Basic formulas of mathematics of finance. Comprehensive treatises such as Moore's "Handbook on Financial Mathematics" develop many formulas for the solution of various problems of finance. Many of these formulas, though convenient, are not necessary. From a strictly theoretical standpoint two principal types of financial problems occur. The first type deals with a single payment exemplified by investments involving *compound interest, compound amounts, present values* and *compound discounts.* The second type deals with a series of equal payments, exemplified by *annuities,* their *amounts* and *present values.*

The first type may be solved by the two formulas,

a) $S = P\,(1 + i)^n$, and *b*) $P = S\,(1 + i)^{-n}$,

and the second by the two formulas,

c) $S_n = R\,\dfrac{(1 + i)^n - 1}{i}$ and *d*) $A_n = R\,\dfrac{1 - (1 + i)^{-n}}{i}$.

In all four of the above formulas:
i = interest rate per conversion period,
n = number of conversion periods.

In formula (*a*):
S = compound amount,
P = original principal invested.

In formula (*b*):

P = the present value of the sum S,

n = periods in advance of the maturity date.

In formula (*c*):

S_n = amount of annuity,

R = periodic rent.

In formula (*d*):

A_n = present value of annuity,

R = periodic rent.

It is to be noted that the expression $(1 + i)^{\pm n}$ is basic in the above formulas. Indeed this expression forms the basis of most of the formulas of mathematics of finance. As illustrated in the following examples, it can be very conveniently evaluated on the DECI-LON slide rule.

Example 1. Find the compound amount of \$2500 if interest is converted semi-annually at 5% for 15 years.

Solution: Here $P = \$2500$, $i = \frac{5\%}{2} = 0.025$, $n = 2 \times 15 = 30$.

Formula (*a*) above, adapted to the given data, yields:

$$S = P\,(1 + i)^n = 2500\,(1.025)^{30}.$$

The value of the expression $(1.025)^{30}$ is found by making the following setting:

push hairline to 1.025 on *Ln1*,
draw left index of *C* under the hairline,
push hairline to 30 on *C*,
under hairline read 2.10 on *Ln2*.
$S = 2500 \times 2.1 = \$5250$.

Example 2. Find the present value of an annuity whose periodic rent is \$200 paid quarterly for 10 years, assuming money to be worth 5% converted quarterly.

Solution: The present value is

$$A_n = R\,\frac{1 - (1 + i)^{-n}}{i}.$$

Here $R = 200$, $n = 40$, and $i = \frac{.05}{4} = 0.0125$.

$$\text{Hence } A_n = 200\,\frac{1 - (1.0125)^{-40}}{0.0125}.$$

The expression $(1.0125)^{-40}$ is evaluated by making the following setting:

push hairline to 1.0125 on *Ln1*,
draw left index of *C* under the hairline,
push hairline to 40 on *C*,
under hairline read 0.608 on *Ln-2*.

$$\text{Therefore } A_n = \frac{200 \times (1-.608)}{0.0125} = \$6270.$$

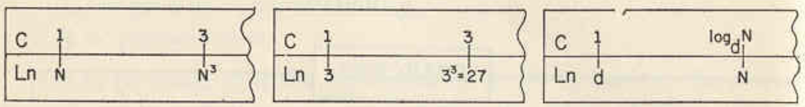
EXERCISES

1. Find the compound amount on \$500 for 20 years at 3% converted semi-annually.
2. To what amount will \$900 accumulate in 15 years, if interest is at 3% converted (*a*) annually? (*b*) semi-annually? (*c*) quarterly? (*d*) monthly?
3. A man borrows \$300, agreeing to repay the principal with interest at 8% converted bi-monthly. What does he owe at the end of 3 years?
4. Find the present value of \$1200 due in 15 years, if money is worth 6%, compounded (*a*) annually, (*b*) semi-annually, (*c*) quarterly, (*d*) monthly.
5. Discount (i.e. find the present value) \$2000 for 20 years if money compounded annually is worth (*a*) 3%, (*b*) 5% (*c*) 6.5%. Find the compound discount in each case.
6. If money is worth 4% converted quarterly, is it more economical to pay \$1000 for a rug now or \$1100 two years hence?
7. Find the amount of an annuity of (*a*) \$200 per annum for 20 years at 6%, (*b*) \$400 each 3 months for 20 years at 6% converted quarterly, (*c*) \$150 each month for 10 years at 6% converted monthly.
8. Find the amount at 7% converted bi-monthly of an annuity of \$200 each half month for (*a*) 2 years, (*b*) 4 years, (*c*) 20 years.
9. A person deposits \$150 every three months in a savings bank which pays 3% converted quarterly. Find the amount of his savings in 10 years.
10. Find the present value of an annuity of (*a*) \$70 a year for 5 years at 6%, (*b*) \$40 each half year for 6 years at 6% converted semi-annually, (*c*) \$10 a month for 10 years at 6% converted monthly.
11. Find the present value of an annuity of \$100 per month at 5% converted monthly for (*a*) 4 years, (*b*) 8 years, (*c*) 12 years.
12. How much must a man deposit in a trust bank on his son's second birthday to provide for his son's college education which he estimates will cost \$1500 a year for 4 years starting on his son's 18th birthday, if the bank's interest rate is 3½%.

72. Logarithms to any base. According to the definition of logarithms, if $d^x = N$, then $x = \mathrm{Log}_d\,N$ (Log of N to the base d). Thus, in the equation $x = \mathrm{Log}_2\,16$, x is the exponent of the power of 2 which equals 16. By inspection, we know that $2^4 = 16$, so $x = 4 = \log_2 16$.

For any number N and any base d (other than 1 or 0), the desired logarithms can be found by using the C scale and the Lon scales in accordance with the relationship established in §70.

The following example and the schematic (Fig. 80) will illustrate the process:



(a) (b) (c)

FIG. 80.

Example. Find $\log_3 27$.

Solution. Let $x = \log_3 27$; then $3^x = 27$. Recall that we can find the power of any number by setting the index of C opposite the number on a Lon scale and using the C scale to multiply by the exponent of the power. In Fig. 80*a*, for example, N on the Lon scale times 3 on the C scale gives N^3 on the Lon scale. In the present example we work with a different unknown (Fig. 80*b*). We locate 3 and 27 on the Lon scale and use the C scale to find the missing exponent. To solve $3^x = 27$,

push the hairline to 3 on *Ln3*,
draw the left index of C under the hairline,
push the hairline to 27 on *Ln3*,
under hairline on C read $3 = x$.

The following rule, illustrated by Fig. 80*c*, will be helpful:

Rule. *To find* $\log_d N$, *push hairline to d on a Lon scale, draw the index of C under the hairline, push hairline to N on a Lon scale, under hairline read* $\log_d N$ *on C. Place the decimal point as described in §70.*

Example. Find (*a*) $\log_3 81$; (*b*) $\log_{0.623} 0.9718$; (*c*) $\log_8 0.726$.

Solution. (*a*) From $L = \log_3 81$, $3^L = 81$. Hence,
push hairline to 3 on *Ln3*,
draw left index of C under the hairline,
push hairline to 81 on *Ln3*,
under hairline read on C, $4 = L$.

The position of the decimal point could have been found by inspection, or by the "Same Scale" principle of §70.

Solution. (*b*) From $L = \log_{0.623} 0.9718$, $0.623^L = 0.9718$. Hence,
push hairline to 0.623 on *Ln-2*,
draw right index of C under the hairline,
push hairline to 0.9718 on *Ln-1*,
under hairline read on C, $0.0605 = \log_{0.623} 0.9718$.

The decimal point was placed in accord with the "Same Scale" and "Tenth Power" principles of §70.

Solution. (*c*) From $L = \log_8 0.726$, $8^L = 0.726$. Hence,
push hairline to 8 on *Ln3*,
draw left index of C under the hairline,
push hairline to 0.726 on *Ln-2*,
under hairline read on C, $-0.154 = L$.

The position of the decimal point was found in accordance with the "Same Scale", the "Tenth Power" and the "Mated Scale" principles of §70.

EXERCISES

Find the value of L in each of the equations:

1. $L = \log_7 100$.
2. $L = \log_{27} 81$.
3. $L = \log_2 32$.
4. $L = \log_{4.4} 0.471$.
5. $L = \log_{0.650} 0.962$.
6. $L = \log_e 1.682$.
7. $L = \log_{3.1} 7.34$.
8. $L = \log_e 10$.
9. $L = \log_{0.2} 0.8$.
10. $L = \log_{1.05} 0.8$.
11. $L = \log_{0.95} 3.47$.
12. $L = \log_{0.93} 25.7$.
13. $\log_L 10 = 2.78$.
14. $\log_L 68 = 0.91$.
15. Find the logarithm to the base 3.34 of each of the following numbers: 42.5, 167, 0.96, 0.267, 0.045.
16. Find the logarithm to the base 0.45 of each of the following numbers 0.682, 50, 100, 0.945.

73. Use of the Lon scales to evaluate logarithms to the base 10. Logarithms to the base 10 are called common logarithms. The values of these logarithms between 1 and 10 are published in tables and are called the *mantissas* of common logarithms. These mantissas are used extensively to perform the processes of multiplication, division, raising numbers to powers and extracting roots. When using these mantissas for computation the logarithms must be put in a special form.

When it is required to obtain the values of the common logarithms for purposes other than as computation tools, the Lon scales are especially serviceable since by using these scales, the common logarithm is evaluated complete with its algebraic sign and decimal point.

For convenience in locating the decimal point in the logarithm, we can make up a set of "Index Power Ranges" for logarithms to the base 10. Set the index of *C* opposite 10 on the *Ln3* scale and read the digits 4-3-5 on the *C* scale opposite the index of the *Ln3* scale. Therefore, as described in §70, read:

On *Ln3*	$10^{0.435}$	to $10^{4.35}$
On *Ln2*	$10^{0.0435}$	to $10^{0.435}$
On *Ln1*	$10^{0.00435}$	to $10^{0.0435}$
On *Ln0*	$10^{0.000435}$	to $10^{0.00435}$
On *Ln-0*	$10^{-0.000435}$	to $10^{-0.00435}$
On *Ln-1*	$10^{-0.00435}$	to $10^{-0.0435}$
On *Ln-2*	$10^{-0.0435}$	to $10^{-0.435}$
On *Ln-3*	$10^{-0.435}$	to $10^{-4.35}$

Since common logarithms are exponents of the base number 10, these "Index Power Ranges" are especially adapted for determining the position of the decimal point in the logarithm.

Example. Find the common logarithms of the following numbers: 4.8, 1.00452, 64.3, 0.684, 0.00220.

Solution. In determining the position of the decimal point and the algebraic sign of each answer we use the "Index Power Ranges" for logarithms to the base 10. In accord with the italicized rule of §72, we make the following setting:

push the hairline to 10 on *Ln3*,
draw right index of *C* under the hairline,
push hairline to 4.8 on *Ln3*,
under hairline read on *C*, **0.681** = $\text{Log}_{10}4.8$,
interchange the indexes of *C*,
push the hairline to 1.00452 on *Ln0*,
under hairline read on *C*, **0.001960** = $\text{Log}_{10}1.00452$,
push hairline to 64.3 on *Ln3*,
under hairline read on *C*, **1.81** = $\text{Log}_{10}64.3$,

push hairline to 0.684 on *Ln-2*,
under hairline read on *C*, **−0.165** = $\text{Log}_{10}0.684$,
push hairline to 0.00220 on *Ln-3*,
under hairline read on *C*, **−2.66** = $\text{Log}_{10}0.00220$.

As an illustration of the importance of evaluating common logarithms consider the formula

$$\alpha = 10 \text{ Log}_{10} \frac{N_1}{N_2}$$

where $\frac{N_1}{N_2}$ is the ratio between two quantities of the same kind possessing the same unit of measure, and α is a number to which is given the name decibel (db) in honor of the famous inventor, Alexander Graham Bell. This formula is widely used in communicating systems to express the ratio of any two amounts of electric or acoustic power in the same units.

In particular, the number of decibels α corresponding to the ratio between two amounts of electrical power P_1 and P_2 is

$$\alpha = 10 \text{ Log}_{10} \frac{P_1}{P_2}. \qquad (a)$$

Also when two voltages E_1 and E_2, or two currents I_1 and I_2, operate in identical impedances, then

$$\alpha_E = 10 \text{ Log}_{10} \frac{(E_1)^2}{(E_2)^2} = 20 \text{ Log}_{10} \frac{E_1}{E_2} \qquad (b)$$

$$\alpha_I = 10 \text{ Log}_{10} \frac{(I_1)^2}{(I_2)^2} = 20 \text{ Log}_{10} \frac{I_1}{I_2} \qquad (c)$$

where α_E and α_I are expressed in decibels.

Moreover, the intensity level α in decibels of a sound of intensity λ in watts/cm² is

$$\alpha_\lambda = 10 \text{ Log}_{10} \frac{\lambda}{\lambda_0} \qquad (d)$$

where λ_0 is the intensity of an arbitrary standard, taken as 10^{-16} watts/cm² at 1000 cycles/sec. This standard is the intensity of a sound that is just audible to the human ear.

Example. The sound on the deck of a carrier at jet take-off has an approximate intensity of 8.9×10^{-4} watts/cm² measured from the reference intensity of the lowest sound audible to the normal ear, which is taken as 10^{-16} watts/cm². Find the intensity level of the noise of jet take-off.

Solution. Referring to formula (*d*) $\lambda = 8.9 \times 10^{-4}$ watts/cm² and $\lambda_0 = 10^{-16}$ watts/cm². Substituting these quantities in (*d*) we obtain in accord with the rules of logarithms (see §61)

$$\alpha_\lambda = 10 \operatorname{Log}_{10} \frac{\lambda}{\lambda_0} = 10 \operatorname{Log}_{10} \frac{8.9 \times 10^{-4}}{10^{-16}} = 10 \operatorname{Log}_{10} (8.9 \times 10^{12}).$$

The value of $\operatorname{Log}_{10} 8.9$ is found from the following setting:

push hairline to 10 on *Ln3*,
draw right index of *C* under the hairline,
push hairline to 8.9 on *Ln3*,
under hairline read 0.95 on *C*.

The decimal point was placed in the answer by noting that 0.95 lies between the "Index Power Range" 0.435 to 4.35 of the *Ln3* scale computed for common logarithms.

$\operatorname{Log}_{10} 10^{12} = 12$, since $\operatorname{Log}_{10} 10 = 1$.

The result therefore is $10(0.95 + 12) = $ **129.5.**

EXERCISES

Find the values of the common logarithms of the following numbers:

1. 3.47.	4. 1.82.	7. 1.0075.
2. 2920.	5. 0.313.	8. 1.0542.
3. 28.7.	6. 0.00031.	9. 0.0051.

10. In measurements of a sound under test, it is desired that the background noise be a certain amount below the level being measured. If the ratio between the intensity of the sound under test to the intensity of the background noise is 11.5 to 1, find the intensity level of the background noise.

11. The rustle of leaves in a gentle breeze has an intensity of 4.95×10^{-15} watts/cm², and the standard reference intensity is 10^{-16} watts/cm². Find the noise level of rustling leaves.

12. The intensities of various sounds referred to a standard intensity of 10^{-16} watts/cm² are:

Source of Sound	*Intensity (watts/cm²)*	*Source of Sound*	*Intensity (watts/cm²)*
Quiet house	9.2×10^{-13}	Very loud thunder	0.752×10^{-4}
Ordinary conversation	0.054×10^{-9}	Threshold of pain	9.0×10^{-5}

Find the intensity levels of the above sounds.

13. Using formula (*b*) above, find the number of decibels corresponding to the following voltage ratios:

(*a*) 0.0131.	(*d*) 7.21.
(*b*) 0.0075.	(*e*) 3.42.
(*c*) 0.682.	(*f*) 1.0075.

14. Using formula (*c*) above find the number of decibels corresponding to the following current ratios:

(*a*) 0.024.	(*c*) 7.43.
(*b*) 0.0072.	(*d*) 10.6.

15. The neper is a unit of measure defined by

$$N = \operatorname{Ln} \frac{E_1}{E_2}$$

where E_1 and E_2 are voltages and N is in nepers.

Find the number of decibels in one neper.

16. Construct the graph of $y = \operatorname{Log}_{10} x$ by plotting the points whose abscissas are $x = 0.01, 1, 10, 20, 30$.

74. Continuous relation of C scales to Lon scales.

A visualization of the continuous relationship of the *C* scale to each group of Lon scales is useful in the process of scale determination and location of the decimal point.

Set the hairline to 1.015 on *Ln1*, then in accord with §70, under the hairline find 1.015^{10} on *Ln2*, 1.015^{100} on *Ln3*, $1.015^{0.1}$ on *Ln0*, $1.015^{-0.1}$ on *Ln-0*, 1.015^{-1} on *Ln-1*, 1.015^{-10} on *Ln-2*, and 1.015^{-100} on *Ln-3*.

Now with the hairline still set in the above position, draw the index of *C* under the hairline and push the hairline to N on *C*. By so doing each exponent is multiplied by N. Hence under this new position of the hairline is found $1.015^{0.1N}$ on *Ln0*, 1.015^{N} on *Ln1*, 1.015^{10N} on *Ln2*, 1.015^{100N} on *Ln3*, $1.015^{-0.1N}$ on *Ln-0*, 1.015^{-N} on *Ln-1*, 1.015^{-10N} on *Ln-2*, and 1.015^{-100N} on *Ln-3*.

Fig. 81 is a schematic illustration of the above discussion.

C	1	N
Ln-3	1.015^{-100}	1.015^{-100N}
Ln-2	1.015^{-10}	1.015^{-10N}
Ln-1	1.015^{-1}	1.015^{-N}
Ln-0	$1.015^{-0.1}$	$1.015^{-0.1N}$
Ln0	$1.015^{0.1}$	$1.015^{0.1N}$
Ln1	1.015^{1}	1.015^{N}
Ln2	1.015^{10}	1.015^{10N}
Ln3	1.015^{100}	1.015^{100N}

Fig. 81.

As shown in Fig. 82, we can imagine the black Lons placed end-to-end in one continuous scale opposite similarly placed red Lons and four lengths of the *C* scale.

If we start with a number on *Ln1*, such as 1.015, the numbers on the *C* scale labeled (*1*) represent exponents from 0.01 to 0.1, those on *C* scale (*2*) represent exponents from 0.1 to 1.0, those on *C* (*3*) from 1.0 to 10, and those on *C* (*4*) from 10 to 100.

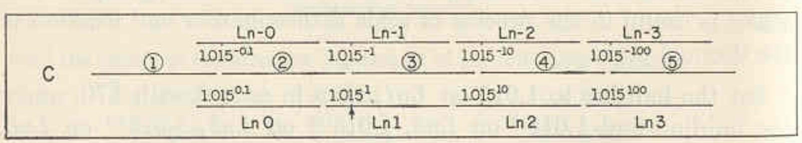


Fig. 82

In general terms, if we set the index of *C* opposite a number *N* on a Lon scale, under the index of *C* on the other Lon scales we find *N* raised to some positive or negative power of 10. Opposite any number *P* on the *C* scale, we find *N* raised to the power of *P* times some positive or negative power of 10.

Example. Evaluate $(1.02)^{2.5}$, $(1.02)^{25}$, $(1.02)^{250}$ and $(1.02)^{0.25}$.

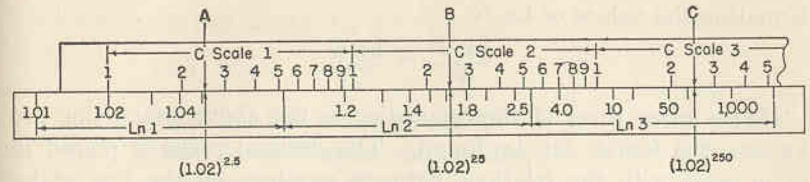


Fig. 83.

Solution: Fig. 83 shows an arrangement in skeleton form of Scales *Ln1*, *Ln2* and *Ln3* placed end-to-end in one continuous scale opposite another continuous scale made up of three *C* scales placed end-to-end. Number these *C* scales from left to right (*1*), (*2*) and (*3*). Referring to Fig. 83, make the following setting:

push hairline to 1.02 on *Ln1*,
draw left index of *C* (*1*) under hairline,

opposite 25 { on *C* (*1*) read on Ln*1*, $1.0508 = 1.02^{2.5}$,
on *C* (*2*) read on Ln*2*, $1.642 = 1.02^{25}$,
on *C* (*3*) read on Ln*3*, $142 = 1.02^{250}$.

By continuing the skeleton leftward so as to include a fourth *C* scale in the *C*-scale chain, say *C* (*0*), while at the same time including the *Ln0* scale in the Lon-scale chain, opposite 25 on *C* (*0*) read $1.00496 = (1.02)^{0.25}$ on *Ln0*.

75. The proportion principle for Lon scales. Fig. 84 indicates a slide rule with an index of the *C* scale set opposite *N* on any Lon scale, *r* on *C* opposite *P* on the Lon scale, and *s* on *C* opposite *Q* on the Lon scale.

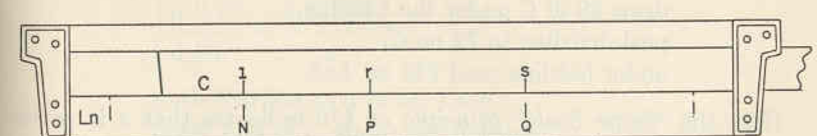


Fig. 84.

Applying the principle of §72, we get from Fig. 84

$$P = N^r, \qquad Q = N^s$$

By taking the logarithm to the base *e* of these two equations and applying logarithmic Law III §61, we get

$$\text{Ln } P = r \text{ Ln } N \qquad \text{Ln } Q = s \text{ Ln } N.$$

Equating the values of Ln N,

$$\frac{\text{Ln } P}{r} = \frac{\text{Ln } Q}{s}.$$

Hence when three of the quantities in the above proportion are known, the fourth can be found. The decimal point is placed in accordance with the relations between numbers on the Lon scales and the C scale set forth in §§72 and 74.

Example 1. Find x in the proportion

$$\begin{array}{ll} \text{on } \mathit{Ln3} \text{ scale:} & \dfrac{\text{Ln } 3.84}{3} = \dfrac{\text{Ln } 9.63}{x}. \\ \text{on } C \text{ scale:} & \end{array}$$

Solution: Push hairline to 3.84 on *Ln3*,
draw 3 of C under the hairline,
push hairline to 9.63 on *Ln3*,
under the hairline read **5.05** on C.

The decimal point in 5.05 was placed in accord with the continuous relation, explained in §74, between the C scale and the *Ln3* scale.

Example 2. Find the value of $x = 8.32^{7.2/2.8}$.

Solution: Equate the Ln's of the two members to obtain

$$\text{Ln } x = \frac{7.2}{2.8} \text{ Ln } 8.32, \text{ or}$$

$$\begin{array}{ll} \text{on } \mathit{Ln3} \text{ scale:} & \dfrac{\text{Ln } x}{7.2} = \dfrac{\text{Ln } 8.32}{2.8}. \\ \text{on } C \text{ scale:} & \end{array}$$

Push hairline to 8.32 on *Ln3*,
draw 28 of C under the hairline,
push hairline to 72 on C,
under hairline read **232** on *Ln3*.

Here the "Same Scale" principle of §70 indicates that x is to be read on *Ln3*.

Example 3. Find x from $x = \sqrt[51]{0.8^{6.4}}$.

Solution: Write x in the form $x = (0.8)^{\frac{6.4}{51}}$ and equate the Ln's of the two members to obtain

$$\text{Ln } x = \frac{6.4}{51} \text{ Ln } 0.8, \text{ or}$$

$$\begin{array}{ll} \text{on Lon scales:} & \dfrac{\text{Ln } x}{6.4} = \dfrac{\text{Ln } 0.8}{51}. \\ \text{on } C \text{ scale:} & \end{array}$$

Push hairline to 0.8 on *Ln-2*,
draw 51 on C under the hairline,
push hairline to 64 on C,
under hairline read **0.9724** on *Ln-1*.

A brief study of Fig. 85 will show the reason for reading the result on *Ln-1*. Whenever the operator is in doubt as to the position of a decimal point or the scale to be used for a reading, he should visualize the continuous relation of C scales to Lon scales as shown in Fig. 85. A rough sketch might be of help.

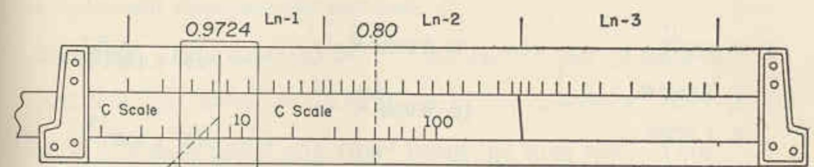


Fig. 85.

Example 4. Solve $0.72^x = 28.7^{1.34}$.

Solution: Equate the Ln's of the two members to obtain

$$x \text{ Ln } 0.72 = 1.34 \text{ Ln } 28.7, \text{ or}$$

$$\frac{\text{Ln } 0.72}{1.34} = \frac{\text{Ln } 28.7}{x},$$

push hairline to 0.72 on *Ln-2*,
draw 1.34 of C under the hairline,
push hairline to 28.7 on *Ln3*,
under hairline read **−13.7** on C.

The minus sign in the answer was determined by noting in the above proportion that Ln 0.72 is negative while Ln 28.7 is positive. This can be seen from the legend numbers of scales *Ln-2* and *Ln-3*. A brief study of Fig. 86 in the light of §74 will indicate the reason for the position of the decimal point in 13.7.

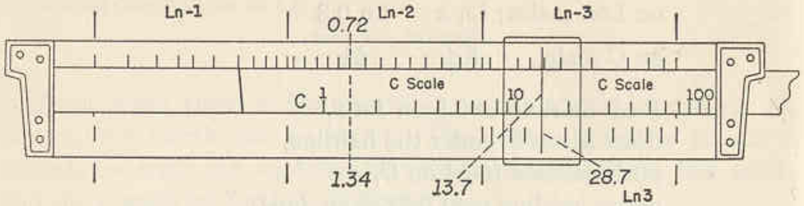


FIG. 86.

EXERCISES

1. $3.82^{\frac{3.82}{1.71}}$.
2. $148^{\frac{72}{342}}$.
3. $7.545^{0.38}$.
4. $1.177^{5.2}$.
5. $1.0733^{\frac{587}{62.9}}$.
6. $0.476^{3/7}$.
7. $0.865^{\frac{42.6}{2.17}}$.
8. $0.865^{\frac{4.26}{2.17}}$.
9. $0.953^{\frac{0.851}{2.64}}$.
10. $0.953^{\frac{8.51}{2.64}}$.
11. $0.953^{\frac{85.1}{2.64}}$.
12. $0.0553^{\frac{2.80}{1.49}}$.
13. $0.0553^{\frac{0.280}{1.49}}$.
14. $0.0553^{\frac{0.280}{14.9}}$.
15. $1.456^{\frac{28.6}{1.172}}$.
16. $1.456^{\frac{2.86}{1.172}}$.
17. $1.456^{\frac{2.86}{11.72}}$.
18. $53.6^{\frac{2.763}{3.07}}$

Solve the following equations for the unknown quantities:

19. $1.14^{0.72} = x$.
20. $4.02^{x} = 8.4$.
21. $12^{y} = 7.137$.
22. $81^{t} = 10$.
23. $y^{2.14} = 4140$.
24. $2.66^{3t} = 12$.
25. $4.02 = (2.37)^{\frac{1}{y+1}}$.
26. $S^{5.32} = 9.96$.
27. $2.37^{\frac{t}{5.3}} = 17.4$.
28. $x^{1.732} = 24.6$.

Solve each equation for x:

29. $0.311^{x} = 10.2$.
30. $5.75^{x} = 0.556$.
31. $1.043^{x} = 0.759$.
32. $0.955^{x} = 25.9$.
33. $x^{-2.14} = 0.617$.
34. $0.435^{x} = 1.475$.
35. $1.056^{-2/x} = 0.623$.
36. $0.054^{-5/x} = 1.355$.

76. Numbers less than 0.00005 or greater than 30000. The methods developed in this article and the next are to be used when the operator, in attempting to solve a problem by previously discussed methods, finds that a required reading involves an extension of the Lon scales. These scales represent numbers from 0.00005 to 30000 except for a gap between 0.999 and 1.001. This article explains methods to be applied for very large numbers or very small numbers; the next article considers numbers between 0.999 and 1.001

The method of attack in finding powers d^m of very large or very small numbers is to write the base d in the powers-of-ten notation, then use the law of exponents (see §61) to resolve the power into several parts, one an integral power of 10 and the others within the range of the rule, and finally make the computation. The following examples will illustrate the method:

Example. Evaluate: (*a*) $24^{5.32}$ (*b*) $247^{5.32}$ (*c*) $(0.0000042)^{2.31}$

Solution: (*a*) $24^{5.32} = (2.4 \times 10)^{5.32} = 2.4^{5.32} \times 10^{.32} \times 10^5$.

Hence, evaluate $2.4^{5.32}$ and $10^{0.32}$ using the slide rule. Thus

to 2.4 on *Ln2* set right index of *C*,
opposite 532 on *C* read on *Ln3*, $105.4 = 2.4^{5.32}$;
to 10 on *Ln3* set left index of *C*,
opposite 32 on *C* read on *Ln2*, $2.09 = 10^{0.32}$.

Therefore, $24^{5.32} = 105.4 \times 2.09 \times 10^5 = 220.5 \times 10^5 = 2.205 \times 10^7$.

(*b*) $247^{5.32} = (2.47 \times 10^2)^{5.32} = 2.47^{5.32} \times 10^{10.64} = 2.47^{5.32} \times 10^{0.64} \times 10^{10} = 122.8 \times 4.37 \times 10^{10} = 536 \times 10^{10} = 5.36 \times 10^{12}$.

(*c*) $(0.0000042)^{2.31} = (4.2 \times 10^{-6})^{2.31} = 4.2^{2.31} \times 10^{-13.86} = 4.2^{2.31} \times 10^{-0.86} \times 10^{-13} = 27.6 \times 0.138 \times 10^{-13} = 3.81 \times 10^{-13}$.

To find the factor $10^{-0.86}$
set left index of *C* to 10 on *Ln3*,
opposite 86 on *C* read on *Ln-3*, $0.138 = 10^{-0.86}$.

EXERCISES

1. 125.6^4. Hint: $(1.256 \times 10^2)^4 = 1.256^4 \times 10^8$.
2. $85.6^{4.32}$. Hint: $(0.856 \times 10^2)^{4.32} = 0.856^{4.32} \times 10^{8.64} = 0.856^{4.32} \times 10^{0.64} \times 10^8$.

3. $0.0001346^{2.65}$. Hint: $(1.346 \times 10^{-4})^{2.65} = 1.346^{2.65} \times 10^{-10.60}$.
4. $0.000894^{0.03}$. Hint: $(8.94 \times 10^{-4})^{0.03} = 8.94^{0.03} \times 10^{-0.12}$.
5. (*a*) $135.8^{4.1}$, (*b*) $78.7^{4.56}$, (*c*) $0.0001257^{2.73}$, (*d*) $0.0008275^{0.032}$.
6. (*a*) $7.84^{6.3}$, (*b*) $9.36^{5.7}$, (*c*) $4.2^{0.0042}$, (*d*) $1.021^{0.32}$.
7. $\log_{10}29{,}300$. Hint: $\log_{10}29{,}300 = \log_{10}293 + \log_{10}100$.
8. Find x if $x^{3.1} = 72{,}000$. Hint: $\dfrac{(x)^{3.1}}{(10)^{3.1}} = \dfrac{72{,}000}{10^{3.1}} = \dfrac{72}{10^{0.1}}$.
9. Find x if $6.4^x = 42{,}000$. Hint: $6.4^{x-2} = \dfrac{42000}{(6.4)^2}$.
10. Find x if $10^x = 580{,}000$.
11. Find x if $5.83^x = 1.005$.
12. The Dutch bought Manhattan Island from the Indians in 1620 for $24. If this sum were invested at compound interest at 3% converted annually to what would it amount to in 1960?

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77. Numbers between 0.999 and 1.001. Imagine a black Lon scale representing numbers ranging from 1.0001 to 1.001 (see Fig. 87) and call it the *Ln00* scale. The calibration marks on this scale, so far as the slide rule is concerned, would coincide with the calibration marks of the *Ln0* scale.

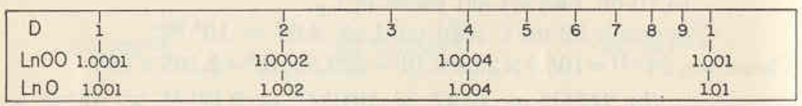


FIG. 87.

The only difference between the two scales would be that the numbers associated with the *Ln00* scale would contain one more significant zero* than is contained in the numbers associated with the *Ln0* scale. Hence, the *Ln0* scale could be used to represent an *Ln00* scale, by mentally adding a significant zero to each reading on the *Ln0* scale. Moreover, a scale, say *Ln000*, dealing with numbers 10 times as close to 1 as those of the *Ln00* scale, could be formed by adding two significant zeroes to each reading on the *Ln0* scale. In this manner, *the Ln0 scale could be used to represent a whole series of Ln scales which might be called Ln00, Ln000, Ln0000,* etc., each scale dealing with numbers 10 times as close to 1 as its predecessor.

* Significant zeros are those to the right of the decimal point, before the first non-zero digit. The number 0.0000506, for example, has four significant zeros.

Similarly imagine a red Lon scale representing numbers ranging from 0.9999 to 0.999 and designate it *Ln-00*. Its calibration marks would coincide with those of the *Ln-0* scale. The numbers represented by these marks in the case of the imaginary scale *Ln-00* would contain one more 9 immediately following the decimal point than the 9's contained in the corresponding numbers of the *Ln-0* scale. Hence, the *Ln-0* scale could be used to represent an *Ln-00* scale by adding a 9 to the 9's immediately following the decimal point contained in the numbers of the *Ln-0* scale. Moreover by adding 9's immediately after the decimal point to the numbers of the *Ln-0* scale, a whole series of red Lon scales could be formed, say *Ln-00*, *Ln-000*, *Ln-0000*, etc., each scale dealing with numbers 10 times as close to 1 as its predecessor.

Note that with this scale designation, for the black Lon group the number opposite the left index of *D* on any scale of this group contains one more significant zero than the number of zeros in the label of that scale. For the red Lon group the number opposite the left index of *D* on any scale of this group contains two more 9's immediately following the decimal point than the number of zeros in the label of that scale.

The following examples will illustrate methods of using the *Ln0* and *Ln-0* scales as imaginary Lon scales dealing with numbers between 0.999 and 1.001.

Example 1. Evaluate:

(*a*) $1.0005^{3.4}$ and 1.0005^{34}; (*b*) $0.9995^{3.4}$ and 0.9995^{-34}.

Solution: (*a*) 1.0005 has three significant zeros; hence it would be on the *Ln00* scale (one scale below the *Ln0* scale). Therefore,

push hairline to 1.005 on *Ln0* (considered as *Ln00*),
draw right index of *C* under the hairline,
push hairline to 34 on *C*,
under hairline read on *Ln0*, **1.001697** $= 1.0005^{3.4}$
and on *Ln1*, **1.01710** $= 1.0005^{34}$.

(*b*) 0.9995 is on the imaginary scale *Ln-00*, one scale above the *Ln-0* scale. Therefore,

push hairline to 0.995 on *Ln-0* (considered as *Ln-00*),
draw right index of *C* under the hairline,
push hairline to 34 on *C*,
under hairline read on *Ln-0*, **0.998295** $= 0.9995^{3.4}$
and on *Ln-1*, **0.98310** $= 0.0005^{34}$.

Example 2. Find the set of values obtained by raising 127 to each of the powers 0.1240, 0.0124, 0.00124, 0.000124, 0.0000124, −0.0124, −0.00124, −0.000124.

Solution: push hairline to 127 on *Ln3*,
draw left index of *C* under the hairline,
push hairline to 124 on *C*,
under hairline read:
on *Ln2*, $1.822 = 127^{0.124}$,
on *Ln1*, $1.0620 = 127^{0.0124}$,
on *Ln0*, $1.00603 = 127^{0.00124}$,
on *Ln00*, $1.000601 = 127^{0.000124}$,
on *Ln000*, $1.0000601 = 127^{0.0000124}$,
on *Ln-1*, $0.9416 = 127^{-0.0124}$,
on *Ln-0*, $0.99401 = 127^{-0.00124}$,
on *Ln-00*, $0.999401 = 127^{-0.000124}$.

EXERCISES

1. Evaluate 1.0002^{22}, 0.9996^{22}, 1.0004^{220}, 1.0006^{220}.
2. Evaluate $9.55^{0.00031}$.
3. Evaluate $0.0455^{0.0052}$, $0.0455^{-0.0052}$, $0.0455^{0.00052}$, $0.0455^{-0.00052}$.
4. Evaluate 1.0006^{5}, 0.9994^{5}, 1.0004^{8}, 0.9996^{8}.
5. Find 1.0009^{45} and 0.9991^{45}.
6. Evaluate 1.00064 and 0.999358 to each of the powers 0.065, 0.65, 6.5, 65.

MISCELLANEOUS EXERCISES

Find the value of unknown quantities represented by x, y, and z in the following equations:

1. $x = 3.15^{2.16}$.
2. $x = 3.15^{0.216}$.
3. $y = e^{-1.74}$.
4. $y = e^{-0.36}$.
5. $x = 0.55^{2.10}$.
6. $z = 0.55^{0.21}$.
7. $x = \sqrt[4]{18.0}$.
8. $x = \sqrt[5]{18.0}$.
9. $y = e^{-0.55}$.
10. $y = 0.35^{0.55}$.
11. $x = e^{1.058}$.
12. $x = e^{-1.058}$.
13. $y = \sqrt[4]{0.698}$.
14. $y = \sqrt[5]{0.645}$.
15. $z = 0.978^{1.80}$.
16. $z = 0.978^{0.180}$.
17. $x = 1.35^{1.92}$.
18. $x = 1.35^{19.2}$.
19. $y = 6.1^{0.48}$.
20. $y = \sqrt[3]{6.1}$.
21. $z^{0.78} = 2.35$.
22. $3.22^{y} = 11.0$.
23. $x^{1.55} = 0.29$.
24. $0.315^{y} = 0.830$.

25. (*a*) What sum would be accumulated in 1000 years if $1,000 were placed at interest @ ¼ of 1% compounded quarterly?

(*b*) If the accumulated sum in (*a*) were then placed at interest of 4% compounded semi-annually for an additional 100 years, how much would be accumulated?

(*c*) If instead of 100 years, the sum in (*b*) were placed at interest for 150 years, how much would be accumulated?

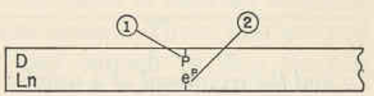
(*d*) If in (*b*) the rate of interest were changed to 5%, how much would be accumulated in 150 years?

(*e*) If in (*b*), the rate of interest remained the same but instead of being compounded semi-annually, was compounded monthly, how much would be accumulated in 150 years?

78. Visual summary.

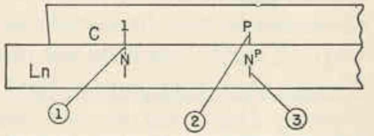
To find the powers of e: $x = e^{P}$

1. Push hairline to P on D,
2. under hairline read e^{P} on Lon scale which contains P between its legend numbers.



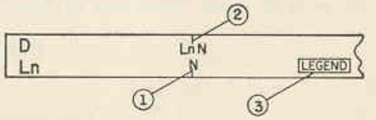
To find the power of any number: $x = N^{P}$

1. Opposite N on a Lon scale set index of C,
2. push hairline to P on C,
3. under hairline read N^{P} on a Lon scale.*



To find the natural log, (log$_e$) of a number: $x = \text{Ln } N$

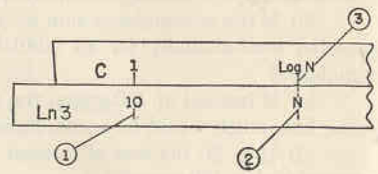
1. Push hairline to N on a Lon scale,
2. read Ln N on D,
3. position decimal point so Ln N lies between legend numbers of Lon scale used.



*For scale selection and location of decimal point see §70.

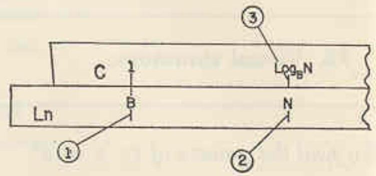
To find the common log (log_{10}) *of a number*: $x = \text{Log } N$

1. Opposite 10 on *Ln 3* set index of *C*,
2. push hairline to *N* on a Lon scale,
3. under hairline read Log *N* on *C*.*



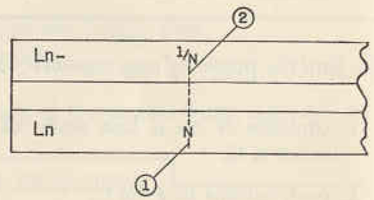
To find a log to any base: $x = \text{Log}_B\ N$

1. Opposite *B* on a Lon scale, set index of *C*,
2. push hairline to *N* on a Lon scale,
3. under hairline read $\text{Log}_B\ N$ on *C*.*



To find the reciprocal of a number: $x = \frac{1}{N}$

1. Push hairline to *N* on a Lon scale
2. read $\frac{1}{N}$ under hairline on mated Lon scale.



* For scale selection and location of decimal point see §70.

CHAPTER VII

THE SLIDE RULE: WHY AND HOW IT WORKS

79. Introduction. Earlier chapters in this manual have explained the slide rule solutions of mathematical problems in terms of rules and settings. They have told *what* to do, without explaining *why* it is done in that manner. For those who find it easier to grasp and remember rules if they understand the principles from which the rules are derived, this chapter explains the basic operating principles which underlie all the scales on the DECI-LON slide rule.

This chapter assumes very little mathematical knowledge on the part of the reader; it starts with an explanation of exponents and logarithms and proceeds through the discussion of each set of scales, ending up with the Lon and Lon minus scales. Those who are well versed in mathematics will find it easy enough to skip the elementary portions.

80. Powers, exponents and logarithms. A number can be multiplied by itself any number of times. For example, $2 \times 2 = 4$, $2 \times 2 \times 2 = 8$, and $2 \times 2 \times 2 \times 2 = 16$.

An *exponent*—a smaller number written above and to the right of the base number—is the mathematical shorthand way of indicating how many times the number appears in the product. Thus 2^2 (pronounced "two squared") $= 2 \times 2 = 4$; 2^3 (pronounced "two cubed") $= 2 \times 2 \times 2 = 8$; 2^4 (pronounced "two to the fourth power") $= 2 \times 2 \times 2 \times 2 = 16$. Similarly, 2^5 (two to the fifth power) is 32 and 2^6 is 64.

The product of two numbers, each representing some power of the same base number, can be obtained by *adding* their exponents. For example: $2^2 \times 2^3 = 2^{2+3} = 2^5$, which is the same as $4 \times 8 = 32$.

If 10 is used as the base number, then

$10^3 = 1000,$

$10^2 = 100$, and, by general agreement,

$10^1 = 10$ (*any* number raised to the exponent 1 is itself),

$10^0 = 1$ (*any* number to an exponent 0 is 1),

$$10^{-1} = \frac{1}{10} = 0.1,$$

$$10^{-2} = \frac{1}{100} = 0.01, \text{ etc.}$$

The exponents are not restricted to integers; they can be fractions as well. For example, $10^{1/2} = \sqrt{10}$ (square root of 10) = 3.16, and $10^{1/4} = \sqrt[4]{10}$ (fourth root of 10) = 1.78. Or if the decimal forms of the fractions are used as exponents, these expressions can be written as follows:

$$10^{0.5} = 3.16, \text{ and } 10^{0.25} = 1.78.$$

Tables are compiled showing the value of various powers of 10, for decimal exponents. A simplified portion of such a table is shown in Fig. 88.

$10^{0.000} = 1$	$10^{0.778} = 6$
$10^{0.301} = 2$	$10^{0.845} = 7$
$10^{0.477} = 3$	$10^{0.903} = 8$
$10^{0.602} = 4$	$10^{0.954} = 9$
$10^{0.699} = 5$	$10^{1.000} = 10$

FIG. 88.

When used in this way, each decimal exponent of 10 is called the *logarithm to the base 10* of the particular number. In the shorthand of mathematics, we write: logarithm of 2 to the base 10 = $\text{Log}_{10}\, 2 = 0.301$; $\text{Log}_{10}\, 3 = 0.477$, etc. Note that

$$\text{Log}_{10}\, 2 = 0.301 \text{ and}$$
$$10^{0.301} = 2$$

are merely two different ways of saying exactly the same thing.

Suppose that the problem is to multiply 2 by 3, but that addition is easier than multiplication. Remembering the rule that two numbers having the same base can be multiplied by adding their exponents, and noting from the above table that $2 = 10^{0.301}$ and $3 = 10^{0.477}$, you can perform the multiplication of 2×3 by writing $10^{0.301} \times 10^{0.477} = 10^{0.301 + 0.477} = 10^{0.778}$. A table of logarithms similar to

Fig. 88 shows that $10^{0.778} = 6$, and thus establishes that $2 \times 3 = 6$. Another way of stating the same problem is: Log 2 + Log 3 = Log 6. Henceforth the designation of the base will be omitted; Log 2 will denote $\text{Log}_{10}\, 2$.

By operating with the logarithms of numbers rather than the numbers themselves, a multiplication problem can be avoided and replaced by a process of addition. Many numerical computations are performed in this manner. Indeed, the technique of multiplying two numbers by adding their logarithms is the basic underlying principle of the slide rule.

81. Multiplication. By using two ordinary rulers, it is possible to *add* two numbers by adding the corresponding lengths on the rulers, as shown in Fig. 89.

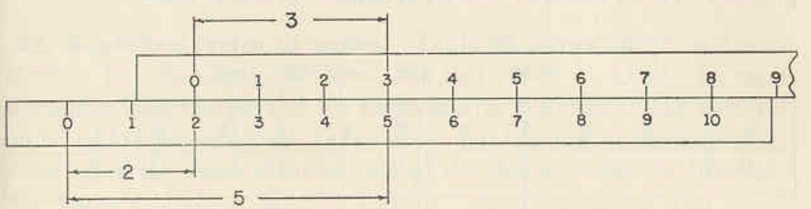


FIG. 89.

On the *C* and *D* scales, the numbers are positioned according to their logarithms. Since the product of two numbers can be obtained by adding their logs, it follows that this product can be found on the slide rule by adding the two logarithmic lengths on the *C* and *D* scales, as shown in Fig. 90.

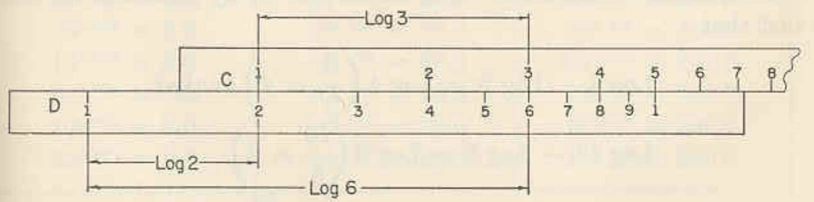


FIG. 90.

The rule for multiplying by means of the slide rule is apparent from Fig. 90. A glance at the figure shows that to multiply two numbers, set the index of *C* opposite one of the numbers on *D*, push the hairline to the other number on C and read the product under the hairline on D.

How is a logarithmic scale, like the D scale, actually constructed? You merely take a scale on which all numbers are equidistant, like an ordinary ruler, and on it locate the logs of the numbers.

The L scale on the slide rule is such a "ruler-type" scale. On it, the distances between any two consecutive calibration marks are equal. The numbers are slightly less than one inch (actually, on a 10-inch rule, 2.5 centimeters) apart. For convenience in measuring lengths proportional to logarithms, the divisions of the L scale are numbered .1, .2, .3, etc., instead of 1, 2, 3.

On the L scale, we locate 0.301—Log 2—and use this distance to locate the primary mark 2 on the D scale. Fig. 91 illustrates the process. The C scale, of course, is similar to the D scale.

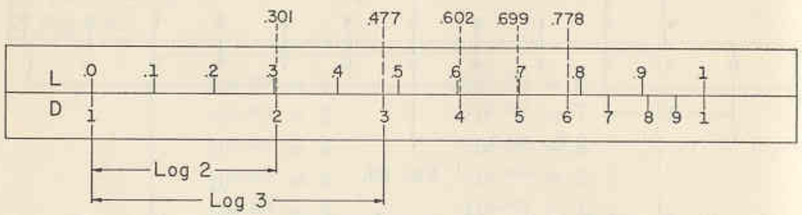


Fig. 91.

82. Division. Just as the product of two numbers is obtained by *adding* their logarithms, the quotient of one number divided by another is obtained by subtracting the logarithm of one number from the logarithm of the other. The reader can verify from Fig. 88 the fact that

$$\text{Log } 8 - \text{Log } 2 = \text{Log } 4 \left(\frac{8}{2} = 4\right) \text{ or that}$$

$$\text{Log } 10 - \text{Log } 5 = \text{Log } 2 \left(\frac{10}{5} = 2\right).$$

In a division problem such as $24 \div 6$, or $\frac{24}{6}$, the 24 is called the *dividend* and the 6 the *divisor*. To divide by means of logarithms, therefore, subtract the log of the divisor from the log of the dividend to get the log of the quotient.

Fig. 92 illustrates schematically the basic rule for division, $P = \frac{M}{N}$, on the slide rule. First push the hairline to the dividend, M, on the D scale, then draw the divisor, N, on the C scale, under the hairline, and opposite the index of the C scale read the quotient, P, on the D scale.

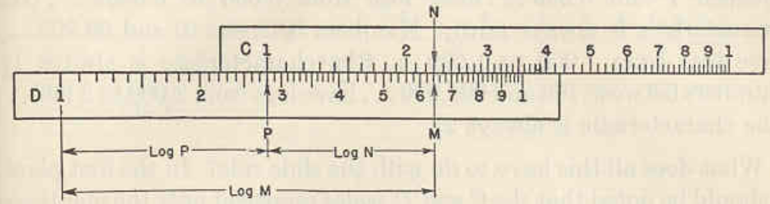


Fig. 92.

83. Repeating cycles of scales. Table 88 shows $10^{0.000} = 1$ and $10^{1.000} = 10$. Similarly, $10^{2.000} = 100$ and $10^{3.000} = 1{,}000$. In each case notice that the part of the logarithm to the *right* of the decimal point—called the *mantissa*—is the same, but the part of the logarithm to the *left* of the decimal point—called the *characteristic*—is different.

Fig. 93 shows the complete logarithms—characteristic and mantissa—for some common numbers.

$10^{0.000} = 1.0$	$10^{1.000} = 10.0$	$10^{2.000} = 100.0$
$10^{0.301} = 2.0$	$10^{1.301} = 20.0$	$10^{2.301} = 200.0$
$10^{0.477} = 3.0$	$10^{1.477} = 30.0$	$10^{2.477} = 300.0$
$10^{0.602} = 4.0$	$10^{1.602} = 40.0$	$10^{2.602} = 400.0$
$10^{0.699} = 5.0$	$10^{1.699} = 50.0$	$10^{2.699} = 500.0$
$10^{0.778} = 6.0$	$10^{1.778} = 60.0$	$10^{2.778} = 600.0$
$10^{0.845} = 7.0$	$10^{1.845} = 70.0$	$10^{2.845} = 700.0$
$10^{0.903} = 8.0$	$10^{1.903} = 80.0$	$10^{2.903} = 800.0$
$10^{0.954} = 9.0$	$10^{1.954} = 90.0$	$10^{2.954} = 900.0$

Fig. 93.

This table clearly illustrates two basic principles which are fundamental to an understanding of logarithms and the slide rule.

First, read across any horizontal row and discover that the mantissas of the logarithms are the same although the characteristics vary. Note also, when 1 is added to the characteristic of a logarithm, the decimal point of the number is shifted one place to the right, and when 1 is subtracted from the characteristic, the decimal point is shifted one place to the left in the number.

Second, read down the vertical columns and find that: numbers between 1 and 9.999 . . . have logs from 0.000 to 0.9999 . . . (the characteristic is always zero). Numbers between 10 and 99.999 . . . have logs from 1.000 to 1.999 . . . (the characteristic is always 1). Numbers between 100 and 999.999 . . . have logs from 2.000 to 2.999 . . . (the characteristic is always 2).

What does all this have to do with the slide rule? In the first place, it should be noted that the *C* and *D* scales represent only the mantissas of the logarithms. That is why 246, or 2.46, or .00246, or 2,460,000 are all located at the same place on the *C* or *D* scale—the logarithms of all these numbers have the same mantissas.

The *D* scale represents *any one* of the vertical columns in a table like that in Fig. 93. The entire realm of numbers is really represented by an endless succession of *D* scales laid end-to-end, with the characteristics added mentally, as shown in Fig. 94.

NUMBERS:	.01–.09	.10–.99	1.0–9.9	10–99.9	100–999.9	1000–9999.9
D SCALES:	———	———	———	———	———	———
LOGS:	$\bar{2}.00$–$\bar{2}.99$	$\bar{1}.00$–$\bar{1}.99$	0.00–0.99	1.00–1.99	2.00–2.99	3.00–3.99

FIG. 94.

Notice that as you move to the left, to encompass numbers smaller than 1, the characteristics are negative. Since $10^{-2.00} = \dfrac{1}{10^2} = 0.01$, it follows that Log 0.01 = $\bar{2}.00$. Notice also that the minus sign is placed above the characteristic and not in front of it, as a reminder that only the characteristic is negative, not the mantissa.

This endlessly repetitive nature of the logarithmic scales explains one principle which usually puzzles slide-rule beginners—namely, the ability to use either index of a scale.

When multiplying 2 by 3 (see Fig. 90), the left index of the *C* scale is used and we seem to be working "to the right". But when multiply-

ing 3 by 4, the right index of the *C* scale is used, and we seem to be working in the opposite direction. Actually, an imaginary *D* scale to the left of the actual scale is used as shown in Fig. 95—the basic operation is the same.

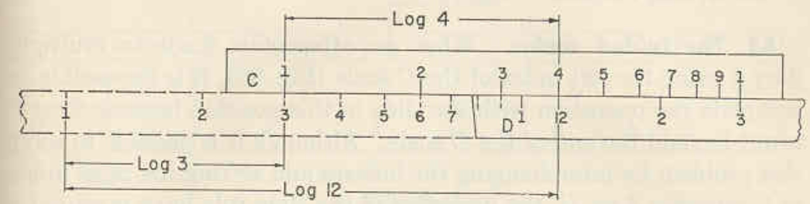


FIG. 95.

Hence, regardless of how small or large the numbers may be, multiplication is accomplished on the slide rule by pushing the hairline to the multiplier on the slide; division by drawing the divisor on the slide under the hairline.

84. Using the *L* scale as a table of logarithms. As demonstrated in Fig. 91, any number *N* on the *C* scale is opposite $\text{Log}_{10} N$ on the *L* scale. Therefore, the *L* scale can be used as a table of logarithms to the base 10. These logarithms are called "common logarithms".

For example, to find Log 450 using the *L* scale push the hairline to 450 on the *C* scale and under the hairline find 653 on the *L* scale. This is the mantissa of Log 450; what is its characteristic? The rules for determining the characteristic are:

1. For logarithms of 1 and numbers greater than 1, the characteristic is one less than the number of digits to the left of the decimal point.

   For example, Logs of 17, 43.487, or 22 will have 1 as a characteristic; Logs of 7, 4.301, or 9.99 will have 0 as a characteristic. In the above example, in accord with Rule (1),

   $$\text{Log } 450 = 2.653.$$

2. For logarithms of numbers less than 1, in the form of decimal fractions, the characteristic is negative and is one more than the number of zeros between the decimal point and the first significant (non-zero) digit.

   For example, Logs of 0.014, 0.08, or 0.07890 will have −2 as a characteristic; Logs of 0.9000, 0.1, or 0.5876 will have -1 as a characteristic.

For practice, the reader can use the *C* and *L* scales to verify that: Log 144 = 2.158; log 12 = 1.079; Log 8.56 = 0.932; Log 0.00306 = $\bar{3}$.486; Log 794,330 = 5.900.

85. The folded scales. When an attempt is made to multiply 2 by 6 using the left index of the *C* scale (Fig. 96), it is impossible to complete the operation with the slide in this position because 6 on *C* is out beyond the end of the *D* scale. Although it is possible to solve this problem by interchanging the indexes and setting the right index of *C* opposite 2 on *D*, the designers of the slide rule have invented a more convenient method.

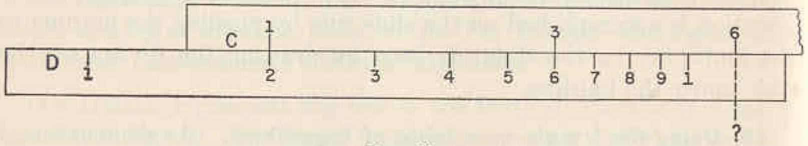


Fig. 96.

Suppose an extra set of *D* and *C* scales is added to the slide rule (Fig. 97), but instead of the left index of each scale occupying its normal position at the left end of the body and slide, each scale is shifted to the right the same distance so that its left index is positioned somewhere near the middle. The righthand portion of each of these scales, which would extend beyond the right end of the slide rule, is simply transplanted, or "folded", around to the lefthand side, as indicated by the dotted lines. These new scales are called the *CF* and *DF* ("*F*" for "folded") scales.

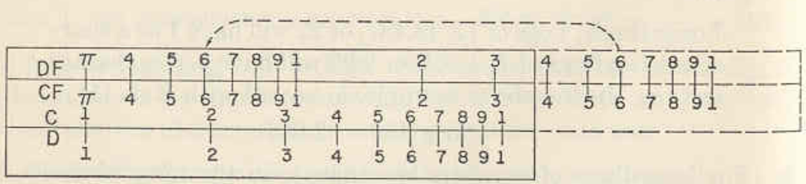


Fig. 97.

Now, regardless of the position of the index of each of these folded scales, any two numbers which are opposite each other in any setting of the *C* and *D* scales will also be opposite each other on the *CF* and

DF scales. Fig. 98 illustrates why this is so. When the index of *C* is set x units to the right of the index of *D*, the index of *CF* is simultaneously set x units to the right of the index of *DF*.

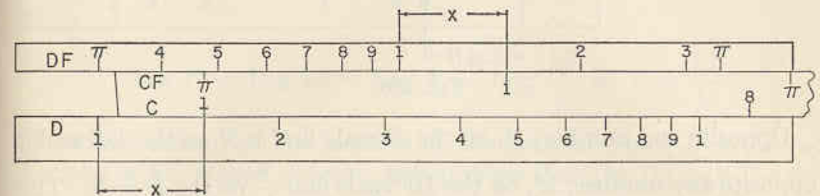


Fig. 98.

If for any setting of the slide it is impossible to read a pair of opposite numbers on the *C* and *D* scales, it will usually be possible to read the required pair on the *CF* and *DF* scales. Fig. 99 shows schematically that the process of multiplying by adding lengths works in exactly the same way on both pairs of scales.

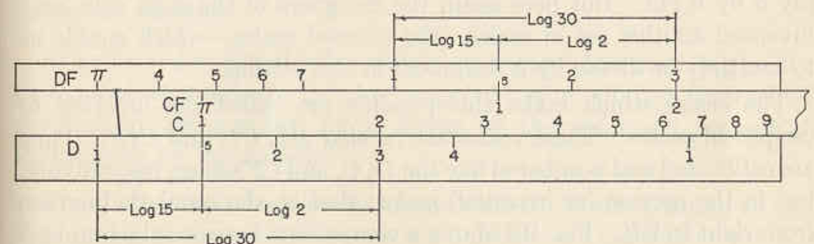


Fig. 99.

These principles apply regardless of the location of the indexes of the folded scales, provided that these indexes are opposite each other when the rule is closed. Since the location of these indexes is arbitrary, and since it is deemed more advantageous to locate them at π, the folded scales are displaced to the left a distance equal to the Log of π. This is shown in Fig. 100, in which the dotted lines represent an imaginary repetition of the *DF* scale. Thus, when the hairline is set to any number *N* on the *D* scale, it is automatically set to Log π + Log *N* on the *DF* scale. Since Log N + Log π = Log $N\pi$, a very effective and convenient way of multiplying by π has been invented.

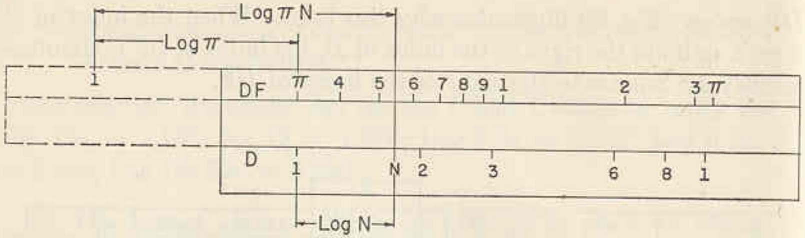


FIG. 100.

Opposite any number, N, on the D scale find πN on the DF scale; opposite any number, M, on the DF scale find $\frac{\pi}{M}$ on the D scale. The same principle, of course, applies to the C and CF scales.

86. The inverted scales. The *reciprocal* of a number is, by definition, 1 divided by the number. Thus the reciprocal of $2 = \frac{1}{2} = 0.5$, the reciprocal of $13 = 1/13 = 0.0769$, and the reciprocal of $a = 1/a$.

We frequently wish to multiply or divide by the reciprocal of a given number. To multiply 5 by the reciprocal of 7, for example, we could use the C and D scales to find that $1/7 = 0.143$, and then multiply 5 by 0.143. But here again the designers of the slide rule have invented another set of scales—the *inverted* scales—which enable us to multiply or divide by a reciprocal in one setting.

The scales which make this possible are called the inverted or reciprocal scales. These scales are labeled DI, CI, and CIF. They are calibrated and numbered like the D, C, and CF scales, respectively, but in the reverse (or inverted) order; that is, the numbers increase from right to left. Fig. 101 shows a comparison between the numbering on the D scale and DI scale.

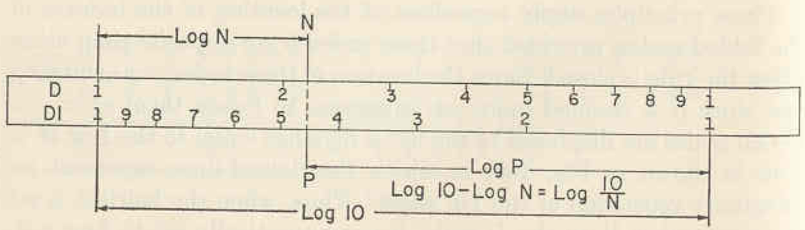


FIG. 101.

Let N designate any calibration mark on the D scale. Then N is

located at a distance equivalent to Log N from the lefthand index of the D scale. Let P be the mark on the DI scale opposite N on the D scale. Since the entire length of each scale is equivalent to Log 10* it follows that

$$\text{Log } N + \text{Log } P = \text{Log } 10.$$

Thus

$$\text{Log } P = \text{Log } 10 - \text{Log } N = \text{Log } \frac{10}{N}, \text{ or } P = \frac{10}{N}.$$

Therefore, except for the position of the decimal point, P is the reciprocal of N. Hence, opposite any number P on the DI scale the reciprocal of P is found on the D scale after placing the decimal point correctly. By a similar process of reasoning it can be shown that opposite any number on D is its reciprocal on DI. The same logic applies, of course, to the C and CI and the CF and CIF scales.

The inverted scales facilitate many continuing series of slide rule operations because they make possible this procedure:

To multiply by the reciprocal of a number, N, push the hairline to N on CI; to divide by the reciprocal of a number N, draw N of CI under the hairline.

As explained in §21, it is frequently convenient and mathematically equivalent to multiply by the reciprocal of a number instead of dividing by the number, or to divide by the reciprocal instead of multiplying by the number.

The CIF scale is both inverted and folded. If the position of the slide makes it impossible to push the hairline to a number on CI, we can often locate it on CIF instead.

87. The *A*, *B*, and *Sq* scales. Before considering how the slide rule deals with squares, square roots, cubes, and cube roots, it may be well to recall the following rule of logarithms and exponents:

$$\text{Log } N^P = P \text{ Log } N.$$

For example, Log $8^4 = 4$ Log 8, Log $17^{14.56} = 14.56$ Log 17, and Log $9^{3/4} = \frac{3}{4}$ Log 9.

*To determine the value of the entire scale length, consider that if we let the left index represent Log 10, the right index represents Log 100; the entire scale therefore represents Log 100 − Log 10 = Log $\frac{100}{10}$ = Log 10. Similarly if we consider the left index as Log 100, the right index represents Log 1000 and the difference is still Log 10.

The *A* and *B* scales are similar to the *C* and *D* scales except that the intervals on *A* and *B* are exactly half as long as those on *C* and *D*.

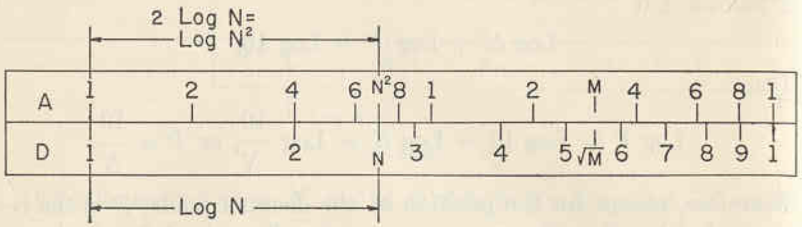


Fig. 102.

A number *N* is located on the *D* scale by measuring off a distance equivalent to Log *N* from the left index of *D*. But since the intervals on the *A* scale are only half as long, that same distance (Fig. 102) represents 2 Log *N*, or Log N^2, on the *A* scale. Therefore, opposite a number *N* on *D* find N^2 on *A*; opposite a number *M* on *A* find $M^{1/2}$ or $\sqrt{M}$ on *D*. A similar statement applies, of course, to the *C* and *B* scales.

When finding a square root by reading from a number on the *A* scale to its opposite on the *D* scale, it is important to determine which half of the *A* scale to use. This table shows why:

N	Log N	Log $\sqrt{N}$ = 1/2 Log N	$\sqrt{N}$
4	0.602	0.301	2
40	1.602	0.801	6.32
400	2.602	1.301	20
4000	3.602	1.801	63.20
40,000	4.602	2.301	200

Note that the Logs of $\sqrt{4}$, $\sqrt{400}$, and $\sqrt{40{,}000}$ have the same mantissa, which is different from the mantissa of the Logs of $\sqrt{40}$ and $\sqrt{4{,}000}$. Therefore the primary number 4 on the left half of the *A* scale can be used for finding the square roots of 4, 400, 40,000, and any number whose logarithm has a characteristic of zero or an even number, but not for 40, 4,000, etc. The primary mark 4 on the right half of the *A* scale is used in finding the square roots of 40, 4,000, 400,000 and any number whose logarithm has an odd character-

istic. The table also illustrates why moving the decimal point two places in *N* is equivalent to moving it one place in $\sqrt{N}$. The use of this principle in locating the decimal point in square roots is explained in §28.

The square scales *Sq1* and *Sq2* are similar to the basic *C* and *D* scales, except that divisions of the square scales are twice the size of divisions on the *C* and *D* scales. Therefore, opposite any number *N* on scale *Sq1* or *Sq2* find its square, N^2, on *D*. Opposite any number *M* on *D* read its square root, $\sqrt{M}$, on *Sq1* or *Sq2*. Here again, in finding square roots, we must determine which square scale to use; the process is similar to that of determining which half of the *A* scale to use, and is explained in §33.

Fig. 103 is a convenient visual summary of the relationships between numbers on the *D*, *A*, and *Sq* scales. Circled numbers indicate the "starting point". Fig. 104 illustrates schematically the relationships among the scales themselves.

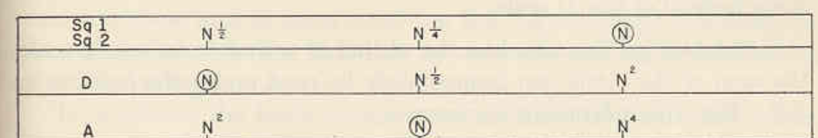


Fig. 103.

Fig. 104.

In addition to using the *A*, *B*, and *Sq* scales for finding the tabular values of squares or square roots, these scales can be used for multiplying or dividing by squares or square roots. The following procedure will become apparent by reference to Fig. 103:

To multiply by $\sqrt{N}$, push the hairline to *N* on *B* (since this is opposite $\sqrt{N}$ on *C*);

To divide by $\sqrt{N}$, draw *N* of *B* under the hairline (since this is equivalent to drawing $\sqrt{N}$ of *C* under the hairline).

Problems requiring multiplication by squares can be handled in one of two ways:

If there is one square in the numerator of a fraction, as in $\dfrac{x^2 \times y}{z}$, start by pushing the hairline to x on *Sq1* or *Sq2*, since this is the same as pushing it to x^2 on *D*. This is the method illustrated in §27.

If a fraction contains several squares, it is often useful to convert the entire expression into a square. Thus

$$\frac{x^2 \times y \times z^2}{p^2 \times q}$$

can be expressed as

$$\left(\frac{x \times \sqrt{y} \times z}{p \times \sqrt{q}}\right)^2.$$

The expression inside the parenthesis can be evaluated by using the *C* and *D* scales for x, z, and p, and the *B* scale for $\sqrt{y}$ and $\sqrt{q}$, and at the end squaring the quotient by reading the answer on the *A* scale instead of the *D* scale.

If the hairline is pushed to the radius of a circle r on an *Sq* scale, the area of the circle can immediately be read under the hairline on *DF*. Fig. 105 indicates the reason.

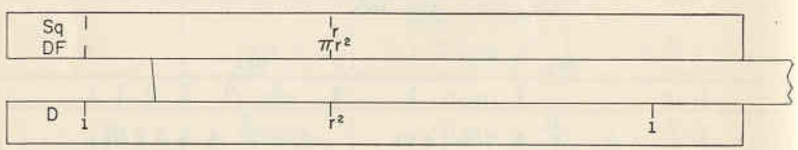


FIG. 105.

88. The *K* scale. The *K* scale is a logarithmic scale with divisions one-third as long as those on the *C* and *D* scales. By a process of reasoning similar to that used in explaining the *A* and *B* scales, it can readily be seen (Fig. 106) that opposite N on *D* is N^3 on *K*, and opposite M on *K* is $\sqrt[3]{M}$ on *D*. Selection of the correct portion of the *K* scale is described in §30.

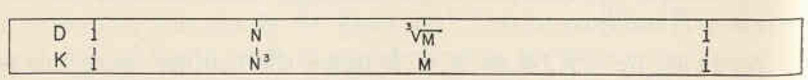


FIG. 106.

89. The trigonometric scales. The three trigonometric scales labeled *S*, *SRT* and *T* are all on the slide. The graduation marks on these scales represent angles and decimal fractions of angles. These scales are a crystallization of many ideas introduced over a long period of time.

The basic tasks required of these scales are that they be, in effect:

(1) a table giving the numerical values of the trigonometric functions, and

(2) a computation device by means of which combined operation problems involving trigonometric functions can be rapidly and simply solved.

Since the numerical values of the trigonometric functions are themselves merely numbers, the design of the trigonometric scales parallels the design of the basic scales *D* and *C*. Thus the intervals between calibration marks on the trigonometric scales are proportional to the logs of the trigonometric functions, just as the intervals on the *C* and *D* scales are proportional to the logs of numbers. Consequently, multiplication and division involving the trigonometric functions can be accomplished on the slide rule by manually adding or subtracting their logarithms.

To illustrate the basic principles of all three trigonometric scales, let us construct the sine scale, *S*, similar to the construction of the basic *C* and *D* scales. The "ruler-type" *L* scale will be used as a yardstick.

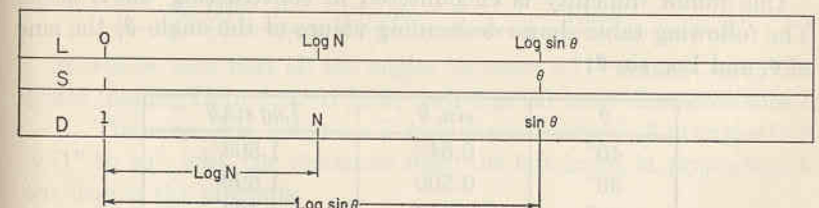


FIG. 107.

Fig. 107 illustrates the relationship between the *L*, *D*, and *S* (sine) scales. To enable us to multiply by N, it was necessary to measure a distance proportional to Log N on the *D* scale. This was done by placing N on the *D* scale opposite the value of Log N on the *L* scale (for example, 3 on the *D* scale is opposite Log 3 = 0.477 on the *L* scale). Similarly, to measure a distance proportional to Log sin θ (the logarithm of the sine of θ) on the S scale, place θ opposite the numerical value of Log sin θ on the L scale. Then (as Fig. 107 shows), just as N on D is opposite Log N on L, similarly sin θ on D is opposite Log sin θ on L, and also opposite θ on S.

As an example, with the slide closed, push the hairline to 30° on the S scale. Under the hairline read 0.5 = sin 30° on D and on the L scale Log sin 30° = Log 0.500 = $\bar{1}.699$.

Now that we have a logarithmic sine scale we can multiply by sin θ by pushing the hairline to θ on S, or divide by sin θ by drawing θ of S under the hairline. Fig. 108 shows schematically how to multiply N by sin θ.

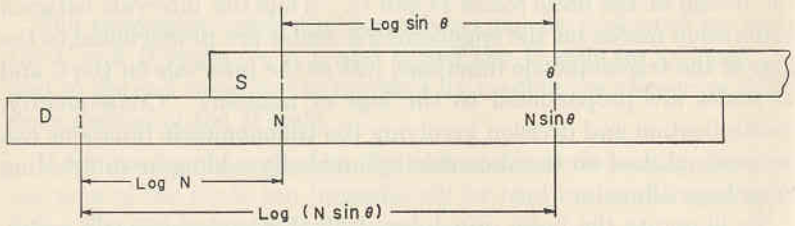


Fig. 108.

One minor difficulty is encountered in constructing the S scale. The following table shows descending values of the angle θ, the sine of θ, and Log sin θ:

θ	*sin* θ	*Log sin* θ
40°	0.643	$\bar{1}.808$
30°	0.500	$\bar{1}.699$
20°	0.342	$\bar{1}.534$
10°	0.174	$\bar{1}.240$
6°	0.104	$\bar{1}.019$
5.7°	0.100	$\bar{1}.000$
5°	0.087	$\bar{2}.940$

Between 6° and 5°, the mantissas run right off the left end of the L scale, because the characteristic changes from −1 to −2. Since

each logarithmic scale on the slide rule represents one complete cycle of mantissas having the same characteristic, the left index of the sine scale must correspond with an angle having a Log sin equal to $\bar{1}.000$ on the L scale, or an angle of approximately 5.7°.

To take care of the smaller angles, another sine scale is constructed for angles whose log sines have a characteristic of −2; that is, angles whose sines range from 0.0100 to 0.0999. These are the angles from 0.57° to 5.7°. For these small angles, the sine, tangent, and value of the angle in radians are all approximately equal. Therefore this same scale can be used for finding tabular values of, and for multiplication by, sine, radians and tangents of small angles. This scale is called the SRT (Sine, Radian, Tangent) scale.

A somewhat different principle is employed in the design of the tangent scale, T. Consider the values of the tangents and the logs of the tangents of the following angles:

Angle	*Tangent*	*Log of tangent*
5.0°	0.087	$\bar{2}.942$
5.71°	0.100	$\bar{1}.000$
20.0°	0.364	$\bar{1}.561$
30.0°	0.577	$\bar{1}.761$
45.0°	1.000	0.000
60.0°	1.732	0.239
70.0°	2.747	0.439
84.29°	10.000	1.000
85.00°	11.430	1.058

It will be seen that all the angles between 5.71° and 45° have tangents ranging from 0.10 to 1.00; their logs all have characteristics of −1. Therefore [Fig. 109(a)] a T scale can be marked off in angles from 5.71° to 45°, with the distances from the left index in proportion to the logs of the tangents.

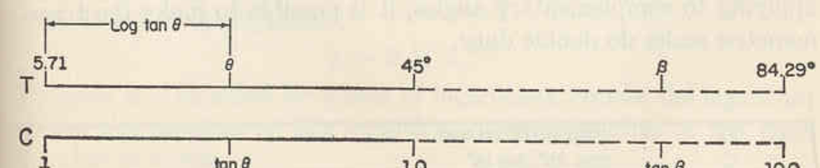


Fig. 109(a).

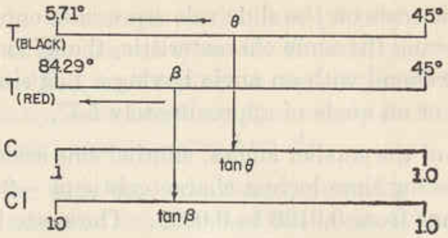


FIG. 109 (*b*)

Angles from 45° to 84.29° have tangents ranging from 1.00 to 10.00; their logs all have characteristics of 0. We could theoretically add another *T* scale and another *C* scale off to the right, as indicated by the dotted lines in Fig. 109(*a*). But we can accomplish the same result in a manner that is more convenient, and at the same time does not require an additional scale.

As shown in Fig. 109(*b*), take the angles from 45° to 84.29° and superimpose them on the *T* scale, except in this case we read from right to left with the figures in red. To read the tangents of these angles we need a basic logarithmic scale reading from right to left—and the *CI* scale is just such a scale.

To find the tangents of angles from 5.71° to 45°, therefore, we use the black figures on the *T* scale and read the tangents on the *C* scale; to find tangents of angles from 45° to 84.29°, we use the red figures on the *T* scale and read the tangents on *CI* on the other side of the rule, or, with the slide closed, on the *DI* scale on the same side as the trigonometric scales. As indicated by the legend at the end of the *T* scale, tangents of the black angles (on *C*) have values from 0.1 to 1.0; tangents of the red angles (on *CI*) have values from 1.0 to 10.0.

The *complement* of an angle is 90° minus the angle. Thus the complement of 30° = 90° − 30° = 60°. By using trigonometric identities applying to complementary angles, it is possible to make the trigonometric scales do double duty.

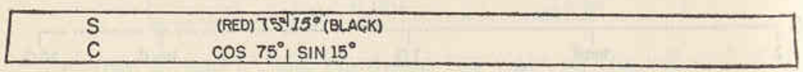


FIG. 110.

From trigonometry recall that $\sin A = \cos (90° - A)$. Thus $\sin 22° = \cos 68°$, $\sin 70° = \cos 20°$, and so on. Each calibration mark on the *S* scale (Fig. 110) represents both an angle and its complement. Opposite each "black" angle is its sine on *C*; opposite each "red" angle is its cosine on *C*.

Similarly, each calibration mark on the *T* scale represents both an angle and its complement. Because $\tan A = \cot (90° - A)$, these relationships apply (Fig. 111):

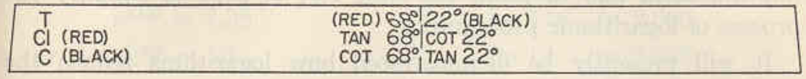


FIG. 111.

Opposite a "black" angle on *T*, read its tangent on the black scale, *C*, and its cotangent on the red scale, *CI*. Opposite a "red" angle on *T*, read its tangent on the red scale, *CI*, and its cotangent on the black scale, *C*.

These rules can easily be remembered by the following summary:

		On T		*On C or CI*
To find a tangent:	read from	black	to	black
	or	red	to	red.
To find a cotangent:	read from	black	to	red
	or	red	to	black.

90. The Lon scales. In §87 it is shown how 8^2 is found by using the *A* scale, and in §88 how 8^3 is found by using the *K* scale. The question naturally arises what happens if some power like $8^{1.7}$ or $17^{-0.062}$ is required.

Before proceeding further let us pause to examine more closely the role played by logarithms in these processes and the corresponding role assumed by the slide rule. Consider, for example, the problem of finding x in the equation:

$$x = 2 \times 3.$$

To solve this equation by means of logarithms, equate the logarithm of the left member of this equation to the logarithm of the right member to obtain

$$\text{Log } x = \text{Log } 2 + \text{Log } 3.$$

The separate values of Log 2 and Log 3 are obtained from a table of logarithms. These values are then added and the table is re-entered to find the number corresponding to this logarithmic sum. This number, called the antilog, is the value of x.

Now the slide rule is an instrument which is capable of mechanically adding logarithms without performing the chore of looking up their separate numerical values in a table, and in addition yields the antilog of the logarithmic sum without recourse to the same table.

So, just as logarithms are employed to simplify the process of multiplication and division, the slide rule is used to simplify the process of logarithmic procedure.

It will presently be demonstrated how logarithms reduce the problem of finding the power of a number first to a problem of multiplication and then to a problem of addition. We shall then see how the slide rule takes over and still further simplifies this logarithmic process.

Now consider the logarithmic procedure for finding $8^{1.7}$. First write

$$x = 8^{1.7} \tag{1}$$

and equate the logarithms of both sides of (1) to obtain

$$\text{Log } x = \text{Log } 8^{1.7} = 1.7 \times \text{Log } 8. \tag{2}$$

The right hand side of (2) involves a multiplication. To perform this multiplication by means of logarithms, equate the logarithm of the left member of (2) to that of the right to obtain

$$\text{logarithm of Log } x = \text{Log } 1.7 + \text{logarithm of Log } 8. \tag{3}$$

In other words, to operate with any power or root of any number, a new kind of scale is needed, on which the distances are proportional *to the logarithms of the logarithms* of numbers.

In the design of all the scales so far considered, logarithms to the base 10, or common logarithms, were employed.

In designing this new scale which involves "logarithms of logarithms" it was found more advantageous to employ a different logarithmic base.

The mathematical constant e is so important in many branches of mathematics and engineering that it is often desirable to work with logarithms to the base e, instead of to the base 10. These

logarithms to the base e are called Napierian logarithms, or natural logarithms. To avoid confusion, let us call the common logarithms "logs" and write them as Log x; and call the natural logarithms "lons" and write them as Ln x.

Before laying out our Lon scales, let us look at some of the powers of e:

$e^{0.693} = 2$	therefore	Ln 2	= 0.693
$e^{1.000} = 2.718$	"	Ln 2.718	= 1.000
$e^{1.609} = 5$	"	Ln 5	= 1.609
$e^{2.000} = 7.38$	"	Ln 7.38	= 2.000
$e^{2.303} = 10$	"	Ln 10	= 2.303.

The table illustrates one important way in which lons differ from logs. Each lon is a complete number, including the decimal point and the digits to the left of it. We can not "unhook" the mantissa, as we can with common logs, because the powers of e do not produce repeating patterns of digits the way the powers of 10 do. There is an obvious relationship between the exponents and the powers in $10^{0.321} = 2.094$, $10^{1.321} = 20.94$, and $10^{2.321} = 209.40$. There is no such pattern in $e^{0.321} = 1.38$, $e^{1.321} = 3.75$, and $e^{2.321} = 10.2$.

By providing a scale on which distances are in proportion to the *common* logarithms of the *natural* logarithms, we accomplish two basic purposes:

1. we can use "logs of lons" to solve equations involving powers or roots, as in equation (3) above; and,

2. by basing this new scale on lons instead of logs, the tabular values of lons and powers of e can be found, and computations involving such numbers can be made.

In laying out the Lon scales the L scale was used as the unit of measure in the same manner as it was used in laying out the C and D scales, except the log of Ln N is located on the L scale instead of the log of number N.

To determine where to place "10" on our Lon scale, for example, we determine that Ln 10 = 2.30259. Log 2.30259 = 0.36222. Therefore (Fig. 112) we place 10 on the Lon scale opposite 0.362 on the L scale. Similarly,

to locate 50: Ln 50 = 3.912 Log 3.912 = 0.5924 on L,
" " 5: Ln 5 = 1.609 Log 1.609 = 0.2066 " ",
" " 3: Ln 3 = 1.099 Log 1.099 = 0.0410 " ".

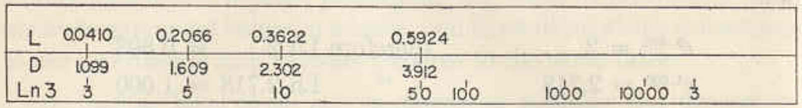


FIG. 112.

The next question is: what numbers will appear on the Lon scale opposite the indexes of the L scale? As Fig. 112 indicates, the common logs on the L scale have characteristics of 0, ranging from 0.00 at the left index to 1.00 at the right index. Therefore (Fig. 113), the left index represents a number whose lon has a log of 0. The lon, therefore, is 1 because $10^0 = 1$. If the lon is 1, the number on the Lon scale must be e itself, since $e^1 = e$, or Ln $e = 1$. Similarly, at the right index we find e^{10}, since this number has a lon of 10, and Log 10 = 1.00.

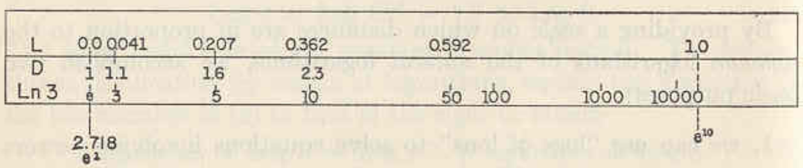


FIG. 113.

Therefore, on the Lon scale just laid out, the primary numbers run from e^1 (2.718) to e^{10} (22,026); and the logs of their lons, laid out on the L scale, have characteristics of 0. Now (Fig. 114) make up three additional Lon scales, place them end to end, and attach them to the left end of the scale just constructed. Place an L scale adjacent to each of these Lon scales. The numbers on the Lon scale next to the original would range from $e^{0.1}$ (1.01) to $e^{1.0}$ (2.718) and the logs of their lons on the adjacent L scale would have characteristics of -1. On the next Lon scale the numbers would range from $e^{0.01}$ to $e^{0.1}$ and the logs of their lons on the adjacent L scale would have characteristics of -2. On the final Lon scale the numbers would

range from $e^{0.001}$ to $e^{0.01}$ and the logs of their lons would have characteristics of -3.

Logs of lons of numbers: L scales:	$\bar{3}.00$ to $\bar{2}.00$	$\bar{2}.00$ to $\bar{1}.00$	$\bar{1}.00$ to 0.00	0.00 to 1.00
Lons of numbers:	0.001 to 0.01	0.01 to 0.1	0.1 to 1.0	1.0 to 10.0
Lon scales: Numbers:	$e^{0.001}$ to $e^{0.01}$	$e^{0.01}$ to $e^{0.1}$	$e^{0.1}$ to $e^{1.0}$	$e^{1.0}$ to e^{10}
Label on Lon scale:	*Ln 0*	*Ln 1*	*Ln 2*	*Ln 3*

FIG. 114.

This is exactly what has been done on the DECI-LON slide rule, except that the four scales, instead of extending off to the left, have been labeled *Ln0*, *Ln1*, *Ln2* and *Ln3* and placed one above the other. The legend at the right of each scale indicates the powers of e represented on that scale. The four red Lon scales, for powers of e with negative exponents, are constructed in a similar manner.

It would hardly be practical to add another Lon scale to the right of *Ln3*, as it would have to include the numbers from e^{10} (22,026) to e^{100} (26,885,000,000,000,000,000,000,000,000,000,000,000,000,000 approximately) and it would be a little difficult to design the scale.

The relationship between the Lon scales and the D scale is illustrated in Fig. 115. Opposite N on a Lon scale is Log Ln N on L, since that is the way we constructed the Lon scale. But just as M on D is opposite Log M on L, in the same way Ln N on D is opposite Log Ln N on L. Therefore, Ln N on D is opposite N on the Lon scale. We can therefore find the lon of a number by pushing the hairline to the number on a Lon scale and reading the lon on the D scale. Fig. 114 verifies the relationship between the legend of the Lon scale (middle row in Fig. 114) and the decimal point in the lon.

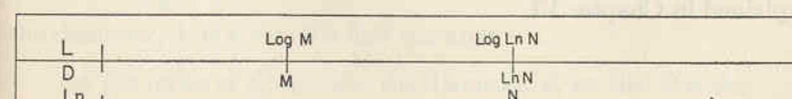


FIG. 115.

Getting back to the problem of finding $x = 8^{1.7}$, we can now take the *lons* (instead of the logs) of both sides of the equation to get:

$$\text{Ln } x = 1.7 \text{ Ln } 8. \qquad (4)$$

Taking the *common* logs of both sides of (4), we have:

$$\text{Log (Ln } x) = \text{Log } 1.7 + \text{Log (Ln } 8). \qquad (5)$$

The setting is shown schematically in Fig. 116. Note that when any equation is put into the form of (5), it is easy to remember that the lons are located on a Lon scale, and the logs on the C or other basic logarithmic scale.

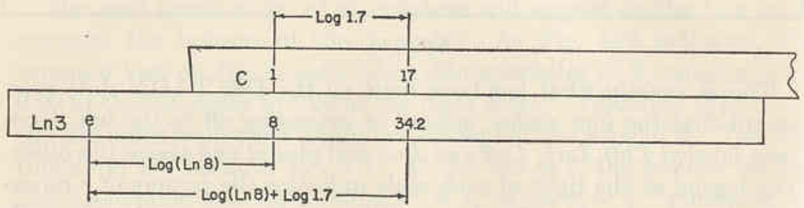


FIG. 116.

Finally, consider the basic relationship between the Lon scales and the D scale (Fig. 117). We have seen that Ln N on D is opposite N on a Lon scale.

Let $x = \text{Ln } N$. That means that

$$e^x = N.$$

Hence in Fig. 117 we can substitute x for Ln N, and e^x for N. This illustrates the function of the Lon scales in finding powers of e.

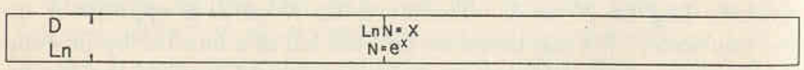


FIG. 117.

The practical applications of these relationships between the Lon scales and the other scales in finding powers, roots, lons, and logs are explained in Chapter VI.

APPENDIX A

LOCATING MARKS FOR BASIC CONSTANTS

The C, D, A, and B scales on the reverse or "red" side of the DECI-LON slide rule include several locating marks to facilitate computations involving frequently used basic constants.

Multiplying or dividing by multiples of π.

On the C and D scales appear the mark "π" at 3.142, the mark "2π" at 6.28, and a tick mark designating $\frac{\pi}{4}$ at 0.785. The use of these marks makes it unnecessary to recall the numerical values when multiplying or dividing by π, 2π, or $\frac{\pi}{4}$, and also insures a more accurate setting.

Multiplying or dividing by $\sqrt{\pi}$ *or* $\sqrt{\frac{\pi}{4}}$.

In combined operations, the values on the A and B scales represent square roots when these scales are used in conjunction with the C and D scales. Pushing the hairline to N on the B scale, for example, is equivalent to pushing it to $\sqrt{N}$ on the C scale, and therefore has the effect of multiplying by $\sqrt{N}$.

The value 3.14 is designated by the mark "π" on the left half of the A and B scales. In combined operations, multiply by $\sqrt{\pi}$ by pushing the hairline to this mark on B, or divide by $\sqrt{\pi}$ by drawing this mark of B under the hairline.

The tick mark just before the 8 on the right half of the A and B scale locates $\frac{\pi}{4}$. This mark is used to find the area of a circle when the diameter, d, is given. To find the area:

set the index of C opposite the diameter, d, on the D scale,

push the hairline to the $\frac{\pi}{4}$ tick mark on B,
under the hairline read the area on A.

This setting is explained by the fact that $A = \frac{\pi}{4}d^2$. Note that the answer is read on the A scale, not the D scale.

An alternative method for finding the area from the diameter, using the *Sq1* and *Sq2* scales, is given in §26.

Finding sines of angles given in minutes or seconds.

For convenience in finding the sines of small angles expressed in seconds or minutes there are two locating marks on the C and D scales on the reverse side of the DECI-LON. The minute mark (′) is just before the 3 (actual value, 0.0002909) and the second mark (″) just before the 5 (actual value, 0.000004848).

To find the sine of an angle expressed in minutes:
set index of slide to the minute mark on D,
opposite any number on C (or CF) representing minutes
read the sine on D (or DF).

The seconds mark is used in the same way. To locate the decimal point it is easy to remember that sin 1″ is approximately "decimal-five-zeros-five" (0.000005) and sin 1′ is approximately "decimal-three-zeros-three" (0.0003).

Converting degrees to radians, or radians to degrees.

The locating mark "R" on the C and D scales on the reverse face of the DECI-LON is used to convert angles from radians to degrees, or degrees to radians. To use it:
set right index of C to "R" mark on D,
opposite any value on C (or CF) in radians read
same angle in degrees on D (or DF);
opposite any value on D (or DF) in degrees read
same angle in radians on C (or CF).

To locate the decimal point recall that 1 radian is approximately 57.3°.

Another method for converting radians into degrees and vice versa using the *SRT* scale is explained in §41.

APPENDIX B

CONVERSION FACTORS

The following tables show a simplified slide rule method of conversion from various units of measurement to others.

For instance: 1 inch = 2.54 centimeters. To convert inches to centimeters, set the index of the C scale to 2.54 on the D scale. Then all readings on the C (CF) scale will represent inches, and the corresponding readings on the D (DF) scale will show the equivalents in centimeters (with proper attention to decimal points).

	Set index of C scale to D scale at:	*On C (CF) scale read measurement in:*	*On D (DF) scale read equivalent in:*
LINEAR MEASURE			
1 inch = 2.54 cm	2.54	in.	cm
1 foot = 0.3048 m	0.3048	ft.	m
1 yard = 0.9144 m	0.9144	yd.	m
1 mile = 1.609 km	1.609	mi.	km
1 mile = 5280 ft.	5280.	mi.	ft.
1 naut. mile = 1.152 mi.	1.152	naut. mi.	mi.
AREA MEASURE			
1 sq. inch = 6.452 cm²	6.452	sq. in.	cm²
1 sq. foot = 0.0929 m²	0.0929	sq. ft.	m²
1 sq. yard = 0.8361 m²	0.8361	sq. yd.	m²
1 sq. mile = 2.59 km²	2.59	sq. mi.	km²
1 sq. mile = 640 acres	640.	sq. mi.	acres
1 acre = 43,560 sq. ft.	43560.	acres	sq. ft.
VOLUME MEASURE			
1 cu. inch = 16.39 cm³	16.39	cu. in.	cm³
1 cu. foot = 0.0283 m³	0.0283	cu. ft.	m³
1 cu. yard = 0.7646 m³	0.7646	cu. yd.	m³
MEASURE OF CAPACITY			
1 U.S. gallon = 3.785 liters	3.785	U.S. gal.	liters
1 U.S. gallon = 231 cu. in.	231.	U.S. gal.	cu. in.
1 cubic foot = 28.32 liters	28.32	cu. ft.	liters
WEIGHT			
1 pound = 0.4536 kg	0.4536	lbs.	kg.
1 grain = 0.0648 g	0.0648	grains	grams
1 U.S. gallon = 8.345 lbs.	8.345*	U.S. gal.	lb.
1 cu. ft. of water = 62.43 lbs.	62.43*	cu. ft.	lb.

**Pure water at maximum density, 39.1° F.*

APPENDIX C

HISTORICAL BACKGROUND

Since logarithms are the foundation on which the slide rule is built, the history of the slide rule rightly begins with John Napier of Merchiston, Scotland, the inventor of logarithms. In 1614 his "Canon of Logarithms" was first published. In presenting his system of Logarithms, Napier sets forth his purpose in these words:

> "Seeing there is nothing (right well beloved Students of Mathematics) that is so trublesome to mathematical practice, nor doth more molest and hinder calculators, than the multiplications, divisions, square and cubical extractions of great numbers, which besides the tedious expense of time are for the most part subject to many slippery errors, I began therefore to consider in my mind by what certain and ready art I might remove those hindrances."

From Napier's early conception of the importance of simplifying mathematical calculations resulted his invention of logarithms. This invention in turn made possible the slide rule as we know it today. Other important milestones in slide rule history follow.

In 1620 Edmund Gunter, of London, invented the straight logarithmic scale, and effected calculation with it by the aid of compasses.

In 1630 William Oughtred, the English mathematician, arranged two Gunter logarithmic scales adapted to slide along each other and kept together by hand. He thus invented the first instrument that could be called a slide rule.

In 1675 Sir Isaac Newton solved the cubic equation by means of three parallel logarthmic scales, and made the first suggestion toward the use of an indicator.

In 1722 John Warner, a London instrument dealer, used square and cube scales.

Circular slide rules and rules with spiral scales were made before 1733, but their inventors are unknown.

In 1775 Thomas Everard, an English Excise Officer, inverted the logarithmic scale and adapted the slide rule to gauging.

In 1815 Peter Roget, an English physician, invented a Log Log scale.

In 1859 Lieutenant Amédée Mannheim, of the French Artillery, invented the present form of the rule that bears his name.

Cylindrical calculators with extra long logarithmic scales were invented by George Fuller, of Belfast, Ireland, in 1878 and Edwin Thacher, of New York, in 1881.

A revolutionary slide rule construction, with scales on both the front and back surfaces of body and slide and with a double faced indicator referring to all scales simultaneously, was patented in 1891, by William Cox, who was mathematical consultant to Keuffel & Esser Co. With the manufacture of Mannheim rules and this new rule, K & E became the first commercial manufacturer of slide rules in the United States. These had previously all been imported from Europe.

Folded scales *CF*, *DF*, and *CIF* were put on slide rules about 1900, to reduce the amount of movement and frequency of resetting the slide. At first the scales were folded at $\sqrt{10}$ but K & E later folded such scales at π so that π could be used as a factor without a resetting. Log Log scales in three sections were put on K & E rules about 1909.

The Parsons invention of about 1919, which included special scales for finding the hypotenuse of a right triangle was incorporated in a rule made in Japan. This rule later included a Gudermannian scale, patented by Okura, enabling the user to read hyperbolic functions.

A scale referring to the *A* or *B* scales to give the logarithms of the co-logarithms of decimal fractions was introduced on K & E slide rules about 1924. Puchstein's scales for hyperbolic functions, patented in 1923, were put on commercial K & E slide rules in 1929. The trigonometric scales were divided into degrees and decimals of a degree, thus making it possible to eliminate all non-decimal sub-divisions from the rule.

K & E introduced a slide rule (patented in 1939) in which all of the trigonometric scales are on the slide and refer to the full length

C and *D* scales. In solving vector problems on this rule or other similar problems involving continuous operations and progressive manipulation, only the final answer needs to be read.

In 1947, on the basis of Bland's invention, the scales of the logarithms of the co-logarithms of decimal fractions were referred to the *C* and *D* scales, correlated with the Log Log scales and also with all of the other scales of the rule, thereby increasing the power of the slide rule by simplifying the solution of exponential or logarithmic problems, the determination of hyperbolic functions, reciprocals, etc.

It was about 1910 when the slide rule first began to come into general use in the United States. In the years that followed, K & E introduced many improvements in the rigidity of frame, indicator design, the precision of graduations, as well as a variety of new scale arrangements. All these have contributed to the wide popularity of the slide rule and its many uses in the mathematics of science and engineering, as well as for calculations of all kinds in business and industry.

Many types of slide rules have been devised and made in small quantities for the particular purposes of individual users. Rules have likewise been made specially for chemistry, surveying, artillery ranging, steam and internal combustion engineering, hydraulics, reinforced concrete work, air conditioning, radio and other special fields. However, the acceptance of such rules has been relatively limited.

The slide rule has a long and distinguished ancestry. The DECI-LON incorporates the most valuable features invented from the beginning of slide rule history with new features to meet modern requirements.

Answers read between 2 and 4 on the *C* scale or *D* scale contain four significant figures, the last one being 0 or 5. Hence such answers have the fourth significant digit accurate to the nearest 5.

§6. Page 8

1. 6	4. 9.1	7. 49.8	10. 0.0826	13. 9.87
2. 7	5. 6.75	8. 340	11. 3220	14. 3.08
3. 10	6. 9.62	9. 47.0	12. 0.836	

§7. Page 9

1. 15	5. 0.001322	9. 244	13. 170.5
2. 15.77	6. 1737	10. 57.1	14. 5620
3. 3530	7. 9.98	11. 0.1621	15. 6890
4. 42.1	8. 1341	12. 0.201	16. 2870

§8. Page 10

1. 2.32	5. 0.000713	9. 1.154	13. 616
2. 165.2	6. 77.5	10. 0.0419	14. 0.0298
3. 0.0767	7. 1861	11. 0.936	15. 4.96
4. 106.1	8. 26.3	12. 1.535	16. 0.332

§9. Page 11

1. (*a*) 1576; (*b*) 2.60; (*c*) 5.25; (*d*) 4.59
2. (*a*) 26.1%; (*b*) 64.4%; (*c*) 220%; (*d*) 2.73%
3. (*a*) 178.9 mi.; (*b*) 121.1 mi.; (*c*) 2140 mi.
4. (*a*) 9.22 yds./sec.; (*b*) 15.02 ft./sec.; (*c*) 186,000 mi./sec.
5. (*a*) 10.13 sec.; (*b*) 29.5 hrs.; (*c*) 323 hrs.

§10. Page 13

1. 36.7	4. 3400	7. 1586	10. 2.35
2. 8.35	5. 0.00357	8. 0.0223	11. 0.0414
3. 0.0000632	6. 13,970	9. 0.01311	12. .0977

§11. Page 14

1. 18.85	4. 1.910	7. 0.275	10. 5.45
2. 26.40	5. 24.9	8. 0.795	11. 8.92
3. 246.0	6. 1.584	9. 6.24	12. 6.25
			13. 86.8

§12. Page 16

1. 25.2	3. 0.1202	5. 11.93	7. 5.07	9. 449.5, 817.5, 992.1, 1435, 1598, 2095, 2328, 2533, 3000
2. 9.27	4. 123.0	6. 45900	8. 54.9	

§16. Page 21

1. $x = 43.3$
2. $x = 169.4$
3. $x = 285$
4. $x = 5.22$
5. $x = 2.30,\ y = 31.8$
6. $x = 51.7,\ y = 3370$
7. $\begin{cases} x = 106.2 \\ y = 30.4 \end{cases}$
8. $\begin{cases} x = 1.586 \\ y = 41.4 \end{cases}$
9. $\begin{cases} x = 0.1013 \\ z = 0.0769 \end{cases}$
10. $\begin{cases} x = 3.97 \\ y = 0.984 \\ z = 0.272 \end{cases}$
11. $\begin{cases} x = 1.315 \\ y = 1.525 \end{cases}$

§16. Page 21

12. $\begin{cases} y = 37.8 \\ y = 69.5 \end{cases}$ 13. $\begin{cases} x = 0.00416 \\ y = 0.0828 \end{cases}$ 14. $\begin{cases} x = 0.1170 \\ y = 0.927 \end{cases}$ 15. $\begin{cases} x = 186 \\ y = 13.42 \\ z = 50.3 \end{cases}$

§17. Page 23

1. 13.71	4. 48.7	7. 42.0	10. 0.1265	13. 9.68
2. 23.0	5. 0.3960	8. 3.14	11. 104.6	14. 47.6
3. 85.0	6. 9.46	9. 3.20	12. 4.07	

§18. Page 24

1. 167.6 cm.
   249 cm.
   980 cm.
2. (*a*) 274 m.
   (*b*) 800 m.
   (*c*) 2.54 m.
3. (*a*) 720 lb.
   (*b*) 2055 lb.
   (*c*) 31.4 lb.
4. (*a*) 235 sq. cm.
   (*b*) 929 sq. cm.
   (*c*) 421 sq. cm.
5. (*a*) 25,700 watts
   (*b*) 3,940,000 watts
   (*c*) 621 watts
6. (*a*) 1.121 gal.
   (*b*) 0.00255 gal.
   (*c*) 0.1504 gal.
7. 29.0 in., 584 in., 62.7 lb. per sq. in.
8. (*a*) 6.12 lb. per sq. in., 7.35 lb. per sq. in., 24.5 lb. per sq. in.
   (*b*) 21.6 cu. in., 33.4 cu. in., 79.9 cu. in., 183.8 cu. in.

§19. Page 26

1. 0.0625, 0.00385, 1.389, 15.39, 0.0575, 0.0541, 0.01490
2. 0.00253, 0.000559, 6.21
3. 2,160
4. 74.0, 10.97
5. 199.5, 8.55
6. See answers to Ex. 2, 3, 4.
7. 2.74, 0.364, 0.392, 2.56, 12.54, 0.0797

§20. Page 28

1. $x = 16.98$, $y = 12.74$
2. $x = 0.0640$, $y = 1.415$
3. $x = 154.9$, $y = 6950$
4. $x = 0.00247$, $y = 645$
5. $\begin{cases} x = 0.0481 \\ y = 0.0435 \\ z = 44.9 \end{cases}$
6. $\begin{cases} x = 11.07 \\ y = 0.0483 \\ z = 0.465 \end{cases}$

§21. Page 30

1. 11.20	7. 17.2	13. 0.00288	19. 3430
2. 2.36	8. 96.1	14. 144,800	20. 4.75
3. 39.4	9. 0.1111	15. 0.0267	21. 0.481
4. 0.001155	10. 150,800	16. 0.279	22. 3.76×10^{-6}
5. 1.512	11. 15.32	17. 41.3	23. 7.05×10^{-7}
6. 1.015	12. 9.76	18. 111.4	24. 2.82
			25. 0.0527

§22. Page 33

1. 2.91	4. 184.4	7. 0.0252	10. 2180
2. 3.55	5. 75.3	8. 7.49	11. 1741
3. 7.83	6. 12.0	9. 17.52	12. 1766

§26. Page 38

1. 625, 1024, 3720, 5620, 7920, 537,000, 204,000, 4.33, 3.07, 0.1116, 0.00001267, 0.908, 27,800,000, 2.24×10^{13}.
2. (*a*) 37.6 ft.^{2}, (*b*) 0.00597 ft.^{2}, (*c*) 965 ft.^{2}, (*d*) 2.35×10^8 ft.^{2}.
3. (*a*) 5.94 ft.^{2}, (*b*) 3510 ft.^{2}, (*c*) 0.445 ft.^{2}, (*d*) 2.76 ft.^{2}.

§27. Page 38

1. 6.14	2. 4.01	3. 0.427	4. 6.68	5. 0.428

§28. Page 40

1. 2.83, 3.46, 4.12, 9.43, 2.98, 29.8, 0.943, 85.3, 0.252, 0.00797, 252, 316
2. (*a*) 231 ft. (*b*) 0.279 ft. (*c*) 5720 ft.
3. (*a*) 18.05 ft. (*b*) 0.992 ft. (*c*) 49.8 ft.

§29. Page 42

1. 24.2	5. 4.43	9. 43.7	13. 109.1
2. 0.416	6. 1.176	10. 29.4	14. 0.0602
3. 8.54	7. 32.8	11. 64.2	15. 1.525×10^5
4. 0.0698	8. 398	12. 11.41	16. 1.589

§30. Page 44

2.06, 3.11, 9.00, 9.47, 19.69, 0.1969, 0.424, 0.914, 44.7, 0.855, 909, 2.15, 4.64, 46.4

§31. Page 45

1. 9.25, 32.8, 238,000, 422,000, 705,000, 3.94×10^8, 0.0925, 29.2, 5.39, 0.0000373, 0.84, 1.464×10^{11}, 5.71×19^{10}
2. 76

§32. Page 46

1. 6.14	2. 4.01	3. 0.427	4. 6.68	5. 9.65

§33. Page 47

1. 8, 13, 9.54, 8.2, 69, 21.8, 0.2688, 0.0875
2. (*a*) 3.658 ft., (*b*) 7.292 ft., (*c*) 5.140 ft., (*d*) 7.820 ft., (*e*) 12.05 ft.

§34. Page 49

1. 2.19	7. 43,100	12. 12.76	17. 5.03
2. 30.9	8. 1.745	13. 76.3	18. 2290
3. 54.3	9. 1.156	14. 2.12	19. 0.0544
4. 0.974	10. 1.193	15. 1.281×10^8	20. 3.29
5. 1.52	11. 90.7	16. 0.00369	21. 0.000867
6. 0.0577			22. 27.3

§38. Page 56

2. (*a*) 0.5	(*b*) 0.616	(*c*) 0.0581	(*d*) 1	(*e*) 0.999
(*f*) 0.0276	(*g*) 0.253	(*h*) 0.381	(*i*) 0.204	(*j*) 0.783
3. (*a*) 0.866	(*b*) 0.788	(*c*) 0.998	(*d*) 0	(*e*) 0.0349
(*f*) 1.00	(*g*) 0.968	(*h*) 0.925	(*i*) 0.979	(*j*) 0.623
4. (*a*) 30°	(*b*) 61°	(*c*) 22°	(*d*) 5.74°	(*e*) 0.86°
(*f*) 38.3°	(*g*) 3.55°	(*h*) 1.775°	(*i*) 66.9°	
5. (*a*) 60°	(*b*) 29°	(*c*) 68°	(*d*) 84.26°	(*e*) 89.14°
(*f*) 51.7°	(*g*) 86.45°	(*h*) 88.22°	(*i*) 23.1°	

§39. Page 58

1. (*a*) $x = 6.10$, $\theta = 61.7°$ (*b*) $\theta = 54°$, $x = 21.6$ (*c*) $x = 30.4$, $y = 44.5$ (*d*) $\theta = 4.92°$, $\phi = 8.08°$
2. (*a*) 2.5 (*b*) 10.39 (*c*) 44 (*d*) 43.9 (*e*) 25.4 (*f*) 44.2 (*g*) 17.69 (*h*) 17.0
3. (*a*) 5.86° (*b*) 55.8° (*c*) 83° (*d*) 59°
4. (*a*) 35.4 (*b*) 80.7 (*c*) 31.9 (*d*) 262
5. (*a*) 0.978 (*b*) 6.02 (*c*) −9.14 (*d*) 16.45

§40. Page 61

1. 0.142, 0.515, 1.907, 0.0177, 3.55, 19.1, 1.09, 7.03, 1.94, 0.524, 56.40, 0.282, 0.0524, 0.918.
2. (*a*) 13.50° (*b*) 38.15° (*c*) 42.60° (*d*) 28.37° (*e*) 3.38° (*f*) 4.69° (*g*) 23.36° (*h*) 2.465° (*i*) 0.855° (*j*) 20.5° (*k*) 74.95° (*l*) 77.91° (*m*) 86.63° (*n*) 45.85° (*o*) 50.95°
3. (*a*) 76.5° (*b*) 51.85° (*c*) 47.40° (*d*) 61.63° (*e*) 86.6° (*f*) 85.3° (*g*) 66.6° (*h*) 87.54° (*i*) 89.145° (*j*) 69.5° (*k*) 15.05° (*l*) 12.09° (*m*) 3.37° (*n*) 44.15° (*o*) 39.05°
4. (*a*) 28.55° (*b*) 24.09° (*c*) 63.4° (*d*) 51.7° (*e*) 50.2° (*f*) 83.14°

§41. Page 64

1. (*a*) 0.0247 (*b*) 0.01454 (*c*) 0.0436 (*d*) 0.0466
2. (*a*) 1.044° (*b*) 2.65° (*c*) 4.96°
3. (*a*) 0.0627 (*b*) 0.000627 (*c*) 0.627 (*d*) 6.27
4. (*a*) 1.696° (*b*) 16.96° (*c*) 0.01696°
5. (*a*) 15.92 (*b*) 7.59 (*c*) 0.00000548 (*d*) 50.6
6. 0.0597, 0.0597, 16.75, 16.75
7. 0.000977, 0.000977, 1023, 1023
8. 0.00436, 229, 229, 0.00436
9. (*a*) 60°, (*b*) 135° (*c*) 2.5° (*d*) 1° (*e*) 150°
10. (*a*) 0.0209 (*b*) 0.0846 (*c*) 5.40 (*d*) 20.0 (*e*) 0.0400 (*f*) 55.5

§43. Page 67

1. 30.5
2. 0.36
3. 4.6
4. 24.2
5. 14.23
6. 16.79
7. 5.26
8. 254
9. 0.0679
10. 0.267
11. 1.35
12. 16.47
13. 2.04
14. 0.720
15. 4.25
16. 1.225
17. 0.0771
18. 0.0963
19. 38.2
20. 0.00319
21. 0.001086
22. 50.9
23. 0.01375
24. 0.0432
25. (*a*) $\theta = 24.94°$, $\alpha = 40.15°$ (*b*) $y = 0.0731$, $\theta = 4.485°$ (*c*) $y = 2580$ (*d*) $y = 25.3$ (*e*) $y = 11.45$ (*f*) $y = 0.0885$ (*g*) $y = 0.638$ (*h*) $\theta = 4.13$ (*i*) 16.43°

§47. Page 73

1. $C = 75°$, $b = 35.45$, $c = 53.3$
2. $C = 55°$, $b = 70.7$, $a = 56.1$
3. $C = 123.2°$, $b = 2255$, $c = 2600$
4. $C = 55.34°$, $b = 568$, $c = 664$
5. $B = 51.33°$, $c = 81.0$, $b = 63.2$
6. $A = 21.17°$, $b = 1884$, $c = 2020$
7. $B = 26°$, $a = 410$, $c = 456$
8. $B = 46.5°$, $a = 7.71$, $b = 8.12$
9. $A = 27.07°$, $a = 24.4$, $c = 53.6$
10. $B = 17.88°$, $b = 26.9$, $c = 87.6$
11. $A = 36.9°$, $B = 53.1°$, $b = 80$
12. $A = 43.3°$, $B = 46.7°$, $b = 0.662$
13. $B = 37.32°$, $a = 5570$, $c = 7010$
14. $B = 34.4°$, $A = 55.6°$, $a = 4.22$
15. $A = 55.3°$, $B = 34.7°$, $a = 740$
16. $A = 60.1°$, $B = 29.9°$, $b = 29.9$
17. $B = 35.3°$, $C = 84.7°$, $c = 138$
18. $A = 87.8°$, $C = 41.1°$, $a = 117$
19. Impossible
20. $B = 28.6°$, $C = 90°$, $b = 4.79$
21. $A = 17.9°$, $C = 53.1°$, $a = 0.076$
22. $b = 279$, $c = 284$, $C = 100.83°$
23. $c = 123.8$, $B = 3.31°$, $C = 116.69°$
24. $B = 4.15°$, $C = 100.85°$, $c = 40.7$
25. $B = 31.35°$, $a = 30.5$, $b = 18.57$
26. $A = 79.15°$, $a = 0.713$, $b = 0.1367$
27. $A = 68.43°$, $b = 0.325$, $c = 0.883$
28. 160.7 yd.
29. 7.07
30. 35.3°
31. 1265 ft.
32. 1029 yds.

§48. Page 75

1. $A = 31.3°$, $B = 58.7°$, $c = 23.7$
2. $A = 41.05°$, $B = 48.95°$, $c = 153.8$
3. $A = 65°$, $B = 25°$, $c = 55.2$
4. $A = 33.15°$, $B = 56.85°$, $c = 499$
5. $A = 39.5°$, $B = 50.5°$, $c = 44$
6. $A = 67.38°$, $B = 22.62°$, $c = 13$
7. $A = 45°$, $B = 45°$, $c = 18.67$
8. $A = 30.6°$, $B = 59.4°$, $c = 82.5$
9. $A = 3.7°$, $B = 86.3°$, $c = 4.8$
10. 50°

§49. Page 77

1. $A = 31.3°$, $B = 58.7°$, $c = 23.7$
2. $A = 41.05°$, $B = 48.95°$, $c = 153.8$
3. $A = 65°$, $B = 25°$, $c = 55.2$
4. $A = 33.15°$, $B = 56.85°$, $c = 499$
5. $A = 39.5°$, $B = 50.5°$, $c = 44$
6. $A = 67.38°$, $B = 22.62°$, $c = 13$
7. $A = 45°$, $B = 45°$, $c = 18.67$
8. $A = 30.6°$, $B = 59.4°$, $c = 82.5$
9. $A = 3.7°$, $B = 86.3°$, $c = 4.8$
10. 50°
11. 18.6, 20°
12. 19.02, 25°

§50. Page 78

1. $A = 119.9°$, $B = 31.1°$, $c = 52.6$
2. $A = 49.05°$, $C = 79.15°$, $b = 104$
3. $A = 55°$, $B = 40.4°$, $c = 285$
4. $B = 39.2°$, $C = 78.8°$, $a = 3.21$
5. $A = 100.95°$, $C = 33.05°$, $b = 19.8$
6. $A = 46.4°$, $C = 6.4°$, $b = 7.43$
7. $A = 121.1°$, $C = 2.4°$, $b = 0.0828$
8. $A = 77.2°$, $B = 43.5°$, $c = 14.99$
9. $B = 13.38°$, $C = 28.32°$, $a = 7420$
10. 10 and 4.68
11. 4.93 mi.
12. Lat. = 218 ft. Dep. = 478 ft.

§51. Page 80

1. $A = 106.77°$, $B = 46.9°$, $C = 26.33°$
2. $A = 27.35°$, $B = 143.1°$, $C = 9.55°$
3. $A = 52.4°$, $B = 59.4°$, $C = 68.2°$
4. $A = 49.2°$, $B = 37.6°$, $C = 93.2°$
5. $A = 106.3°$, $B = 37.9°$, $C = 35.8°$
6. $A = 48.2°$, $B = 58.4°$, $C = 73.4°$
7. $A = 44.6°$, $B = 49.5°$, $C = 85.9°$
8. $A = 83.7°$, $B = 59.3°$, $C = 36.9°$
9. $A = 53.1°$, $B = 59.5°$, $C = 67.4°$
10. 51.9°, 59.4°, 68.7°
11. 72.58°

§52. Page 82

1. $B_1 = 66.1°$, $C_1 = 58.5°$, $c_1 = 18.6$, $B_2 = 113.9°$, $C_2 = 10.7°$, $c_2 = 4.08$
2. $B_1 = 16.72°$, $A_1 = 147.46°$, $a_1 = 35.5$, $B_2 = 163.28°$, $A_2 = 0.9°$, $a_2 = 1.04$
3. $A_1 = 70.3°$, $B_1 = 57.3°$, $b_1 = 28.8$, $A_2 = 109.7°$, $B_2 = 17.9°$, $b_2 = 10.51$
4. $A_1 = 69°$, $C_1 = 67°$, $a_1 = 6.93$, $A_2 = 23°$, $C_2 = 113°$, $a_2 = 2.91$
5. $B_1 = 45.3°$, $C_1 = 99.1°$, $c_1 = 300$, $B_2 = 134.7°$, $C_2 = 9.7°$, $c_2 = 51.1$
6. $A_1 = 51.3°$, $C_1 = 88.7°$, $c_1 = 21{,}900$, $A_2 = 128.7°$, $C_2 = 11.3°$, $c_2 = 4290$
7. $p = 3.13$; (*a*) none, (*b*) 2, (*c*) 1

§54. Page 86

1. 677
2. 173.4
3. 129.4
4. 415
5. 41.7
6. 376
7. 382
8. 89.3 ft.
9. 10,910 ft.
10. 284 ft., 291 ft.
11. 864 ft., 708 ft., 246 ft.
12. 7.87 mi.

§55. Page 90

1. $x = 35.8$, $y = 19.36$; $r = 32.4$, $\theta = 55.6°$; $w = 18.0$, $\theta = 43.2°$
2. 16, 20°
3. 20, 25°
4. 2.4, 0.7
5. $x = 10.79$ lb., $y = 26.7$ lb.
6. $r = 7.81$, $\theta = 74.2°$
7. 4.5, 126.9°
8. 206.6°
9. $m = 8.94$, $n = 4.47$
10. $z = 9.25$, $\theta = 27°$, $G = 0.0963$, $B = 0.0491$
11. $z = 14.42$, $B = 0.0551$, $G = 0.0422$

§57. Page 93

1. (*a*) 42.3° (*b*) 69.2° (*c*) 92.35°
2. (*a*) 70.5° (*b*) 54.7° (*c*) 58.7°
3. $c = 157.5°$, $A = 101.05°$, $B = 114.6°$
4. (*a*) $A = 129.6°$ (*b*) $A = 148°$

§58. Page 95

1. (*a*) 64.15° and 115.85° (*b*) 137.4°
3. $c = 157.6°$, $A = 78.5°$, $B = 114.7°$
4. (*a*) $d = 1020$ nautical miles, $C =$ N 40.7° W (*b*) 618 nautical miles, $C =$ N 78.8° W

§61. Page 99

1. $4x^6y^{-10}$
2. $1/81\ x^{-16}y^8$
3. $8x^6y^{-9}$
4. $16c^8d^{-12}$
5. $-4^{1/4}c^{1/2}d^{-10/3}$
6. $x^{-2} - 2x^{-1} + 1$

§64. Page 104

1. 6.21, 3.91, 0.693, 0.3365, 0.0421
2. −6.21, −3.91, −0.693, −0.3365, −0.0953, −0.01975
3. (*a*) 4.33 (*b*) 2.03 (*c*) 2.22 (*d*) −0.1744 (*e*) −1.93 (*f*) −0.1132 (*g*) −0.0954 (*h*) 0.358 (*i*) 0.0421
5. 1.386, 0.25, 1.1488, 0.8705, 1.01396, 0.98623, 1.001386, 0.998614.
6. 24.1 yr.
7. 64.6 days
8. 1.356 hr.
9. 11.55 yr.
10. 0.742 neper.

§65. Page 106

1. (*a*) 20.1 (*b*) 0.0498 (*c*) 1.492 (*d*) 0.670 (*e*) 1.03562 (*f*) 0.9656 (*g*) 3.827 (*h*) 0.2613 (*i*) 0.0854 (*j*) 34.8 (*k*) 0.9740 (*l*) 1.0890

§65. Page 106

2. (*a*) 8.33 (*b*) 0.1200 (*c*) 1.2361 (*d*) 0.8090 (*e*) 1.02143 (*f*) 0.9790
3. (*a*) 54.6 (*b*) 3640 (*c*) 1.537 (*d*) 1.0216 (*e*) 0.0334 (*f*) 0.8496 (*g*) 0.9817 (*h*) 0.00203
4. (*a*) 2.028 (*b*) 0.493 (*c*) 0.6553 (*d*) 1.526 (*e*) 2.314 (*f*) 0.4321 (*g*) 9.36 (*h*) 0.1069 (*i*) 0.686 (*j*) 1.458 (*k*) 1.781 (*l*) 0.561 (*m*) 25.0 (*n*) 25.95 (*o*) 2.201
5. 17.33 sec.
6. 29.69 lb.

§66. Page 108

1. 1.238
2. 6.25
3. 0.887
4. 0.822
5. 1.155
6. 0.544
7. $L = 129.2$ ft., $s = 53.0$ ft.
8. $L = 594$ ft., $s = 98.4$ ft.

§67. Page 109

1. (*a*) 5.86×10^3 (*b*) 6.75×10^5 (*c*) 6.23×10^{-2} (*d*) 4.79×10^{-1} (*e*) 2.8×10^7 (*f*) 2.76×10^9 (*g*) 9.1×10^{-7} (*h*) 4.95×10^3 (*i*) 8.645×10^{-3}
2. (*a*) 1.8×10^7 (*b*) 3.43×10^{-7} (*c*) 2.7×10^{-5} (*d*) 2.45×10^4 (*e*) 8×10^4 (*f*) 2×10^{-4}
3. (*a*) 6.91×10^{-7} (*b*) 9.04×10^9 (*c*) 8.48×10^4 (*d*) 2.37×10^3 (*e*) 1.288×10^1
4. (*a*) 4.31×10^{-1} (*b*) 1.17×10^3 (*c*) 1.59×10^4 (*d*) 3.8 (*e*) 2.38×10^1 (*f*) 3.44×10^2
5. 5.88×10^{12} mi.

§68. Page 110

1. 0.2, 0.8, 0.9615
2. (*a*) 0.0625 (*b*) 0.284 (*c*) 64.5 (*d*) 0.513 (*e*) 1.330 (*f*) 0.860 (*g*) 0.9860 (*h*) 1.0510 (*i*) 0.99255

§69. Page 112

1. 4, 25, 49, 64, 0.25, 0.04, 0.0204, 0.015625
2. 0.7646, 0.626, 0.405, 1.308, 1.598, 2.47
3. 0.3333, 1.1161, 0.8960, 1.01105, 0.98908
4. 9, 0.1111, 16, 0.0625, 36, 0.0278, 81, 0.0123, 166.4, 0.00601
5. 0.9434, 1.791, 0.5584, 339, 0.00295
6. 10, 0.7943, 1.259, 0.97724, 1.0233
7. 5, 0.2, 1.1746, 0.8513
8. 0.5927, 1.687, 0.00535, 187, 0.9490, 1.0537

§70(a). Page 114

1. 44.7, 730, 20000
2. 5.15, 3.60, 2.86
3. 1.988, 2.62, 1.1063, 1.1666, 1.283, 1.291
4. 1.0339, 1.078, 1.096, 1.0202, 1.0164, 1.01055
5. 1.00983, 1.00692, 1.00475, 1.00308, 1.00159, 1.001064

§70(b). Page 116

1. 44.7, 1.462, 1.0387, 1.00381
2. 1.00382, 1.0388, 1.464
3. 1.1043, 1.00997, 2.70
4. 2.55, 1.0983, 1.00941, 11700
5. 1.0331, 8000, 8.22

§70(c). Page 116

1. 260, 0.00381, 3700, 0.000270, 0.00129
2. 1.0896, 1.00862, 0.99144, 0.691
3. 0.9688, 0.9747, 0.684
4. 1.487, 22.5, 0.5553
5. 1.0281, 1.346, 1.00507

§70(d). Page 120

1. 1.0474, 1.589, 0.00975
2. (*a*) 1.0550, 0.9479, 0.5857 (*b*) 48.6, 0.0206, 1.0396 (*c*) 0.576, 0.00403, 1.0567 (*d*) 0.283, 1.01269, 3.53
3. 1.707
4. 2.534
5. 1.475
6. 0.9646
7. 0.0273
8. 36.7
9. 11.4
10. 0.0879
11. 1.1874
12. 1.01736
13. 0.842
14. 0.0492
15. 2.380
16. 0.8205
17. 0.9437
18. 329
19. 0.9707
20. 0.0532
21. 3.00
22. 0.333
23. 4.14
24. 0.242
25. 2.23
26. 0.9365
27. 0.0764
28. 1.2934
29. 3.893

§71. Page 123

1. \$907
2. (*a*) \$1402 (*b*) \$1407 (*c*) \$1409 (*d*) \$1410
3. \$381
4. (*a*) \$500 (*b*) \$494 (*c*) \$491 (*d*) \$488
5. (*a*) \$1107 (*b*) \$754 (*c*) \$568, \$892, \$1246, \$1432
6. Pay for rug now
7. (*a*) \$7360 (*b*) \$61,080 (*c*) \$24,600
8. (*a*) \$10,280 (*b*) \$22,120 (*c*) \$209,000
9. \$6960
10. (*a*) \$295 (*b*) \$398 (*c*) \$901
11. (*a*) \$4340 (*b*) \$7900 (*c*) \$10,810
12. \$3180

§72. Page 125

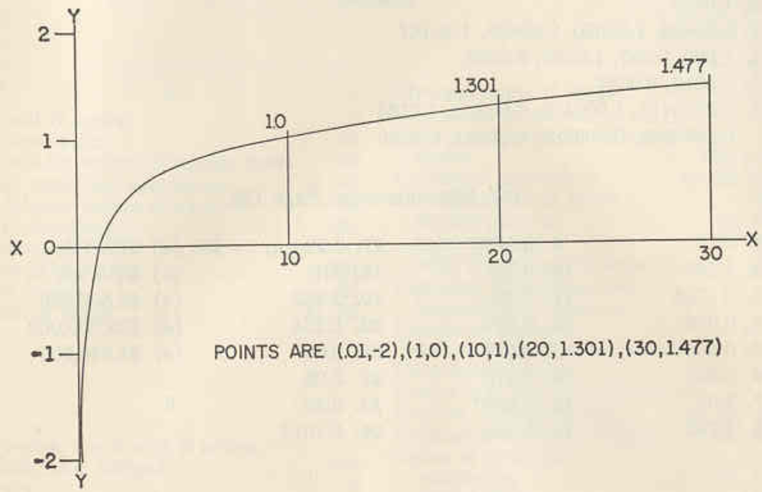
1. 2.365
2. 1.333
3. 5
4. −0.508
5. 0.0899
6. 0.52
7. 1.762
8. 2.303
9. 0.1386
10. −4.57
11. −24.3
12. −44.7
13. 2.29
14. 103.2
15. 3.11, 4.24, −0.0339, −1.095, −2.57
16. 0.479, −4.9, −5.77, 0.0708

§73. Page 128

1. 0.540
2. 3.47
3. 1.458
4. 0.260
5. −0.505
6. −3.509
7. 0.00324
8. 0.02292
9. −2.292
10. 10.61 (db) below sound being tested
11. 16.95 (db)
12. Quiet house 39.64 db., Very loud thunder 118.76 db.
    Ordinary conversation 57.3 db, Threshold of pain 119.54 db.
13. (*a*) −37.66 db. (*c*) −3.32 db. (*e*) 10.68 db.
    (*b*) −42.5 db. (*d*) 17.16 db. (*f*) 0.0652 db.
14. (*a*) −32.4 db. (*b*) −42.84 db. (*c*) 17.42 db. (*d*) 20.5 db.
15. 8.686 db.

§73. Page 128

16.



§75. Page 134

1. 20
2. 2.864
3. 2.155
4. 2.335
5. 1.935
6. 0.7275
7. 0.058
8. 0.752
9. 0.9846
10. 0.856
11. 0.2119
12. 0.00435
13. 0.5805
14. 0.9471
15. 9600
16. 2.5
17. 1.096
18. 36
19. 1.0989
20. 1.53
21. 0.791
22. 0.524
23. 49
24. 0.847
25. −0.379
26. 1.541
27. 17.54
28. 6.35
29. −1.988
30. −0.3356
31. −6.55
32. −70.7
33. 1.253
34. −0.467
35. 0.23
36. 48

§76. Page 135

1. 2.49×10^8
2. 2.23×10^8
3. 5.52×10^{-11}
4. 0.81
5. (*a*) 5.56×10^8
   (*b*) 4.42×10^8
   (*c*) 2.24×10^{-11}
   (*d*) 0.797
6. (*a*) 4.31×10^5
   (*b*) 3.44×10^5
   (*c*) 1.0061
   (*d*) 1.0067
7. 4.467
8. 57.19
9. 5.74
10. 5.76
11. 0.00283
12. \$557,000

§77. Page 138

1. 1.00440, 0.9912, 1.092, 1.1414
2. 1.0007
3. 0.98404, 1.01620, 0.99839, 1.00161
4. 1.003, 0.997, 1.0032, 0.9968
5. 1.0413, 0.9604
6. 1.0000416, 1.000416, 1.00416, 1.0424
   0.9999582, 0.999582, 0.99582, 0.9590

§77. Miscellaneous. Page 138

1. 11.9	9. 0.5769	17. 1.78	25. (a) $12,100
2. 1.281	10. 0.561	18. 318	(b) $635,000
3. 0.1755	11. 2.88	19. 2.382	(c) $4,600,000
4. 0.698	12. 0.347	20. 1.254	(d) $20,000,000
5. 0.285	13. 0.914	21. 2.99	(e) $4,840,000
6. 0.882	14. 0.916	22. 2.05	
7. 2.06	15. 0.9607	23. 0.45	
8. 1.783	16. 0.996	24. 0.1613	

INDEX

THIRD PRINTING - 14571