Self-teaching
INSTRUCTION MANUAL

DIETZGEN
EVERYTHING FOR DRAFTING SURVEYING & PRINTMAKING

NO. 1725

Vector TYPE Log Log SLIDE RULE

Written by
OVID W. ESHBACH, Dean
H. LOREN THOMPSON
Technological Institute
NORTHWESTERN UNIVERSITY

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DIETZGEN
MANIPHASE MULTIPLEX
(TRADE MARK)

VECTOR TYPE LOG LOG

SLIDE RULE

Manual No. 1786-25

by

VERNON G. LIPPITT,
H. LOREN THOMPSON
and
OVID W. ESHBACH, *Dean*

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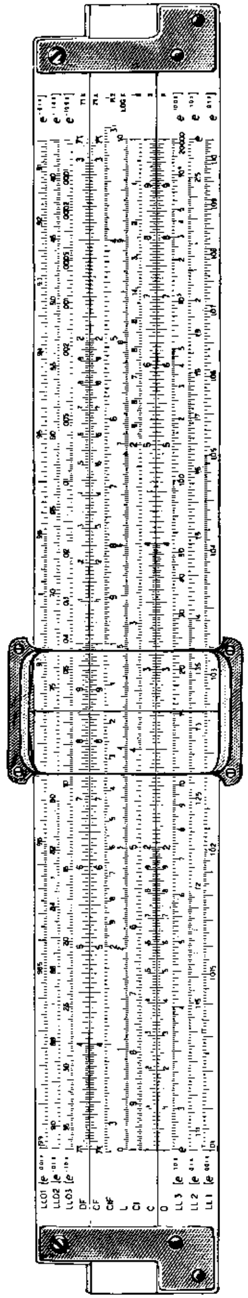

FRONT FACE

THE
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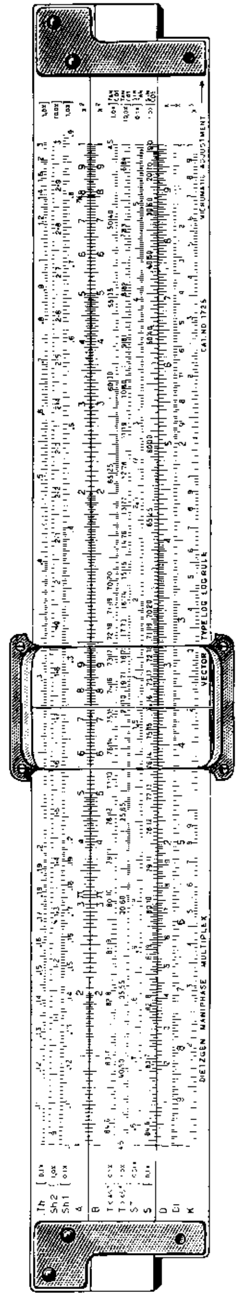


REVERSE FACE

PREFACE

To the beginner, even the simplest slide rule may appear very complex and difficult of mastery. Such a viewpoint should be avoided because it is incorrect and because it is a real handicap in learning to use a slide rule. The mistaken notion that a good slide rule is complex is due to the fact that it is equipped with a multiplicity of scales so that it may be used to solve a wide range of problems.

Actually, a slide rule would be a good investment as a timesaver if it had nothing more than the two scales "C" and "D" for the processes of multiplication and division. And this is exactly the proper starting point for learning how to use a slide rule. Until the "C" and "D" scales are mastered, all other scales are completely ignored. Anyone, with nothing more than a background of simple arithmetic, can learn to multiply and divide by the use of the "C" and "D" scales in short order.

One step at a time this self-instruction manual makes clear the purposes of all other scales on the rule and the manner in which problems involving powers, roots, proportions, trigonometrical functions, logarithmical functions and combinations of these various mathematical considerations may be solved.

It is important to note *all* slide rules from the simplest to the most expensive are based on the same fundamentals—that any problem which can be solved on a simple rule is solved in the same manner or even more simply on a better rule. The more expensive rules merely provide *more* scales for solving problems the less expensive rules cannot handle. This is an important consideration in selecting a slide rule . . . because a rule should be purchased with its ultimate use in mind, rather than the extent of the buyer's mathematical training at the time the purchase is made. As slide rules are normally purchased for a lifetime of use, the more expensive rules are usually the best investments as they not only provide the finest in materials and precision construction, but a range of usefulness always adequate for the needs of their owners.

Eugene Dietzgen Co.

CONTENTS

CHAPTER I

GENERAL THEORY AND CONSTRUCTION

1. The Slide Rule is a tool for rapidly making calculations. It is an indispensable aid to the engineer and the scientist as well as to the accountant, statistician, manufacturer, teacher, and student or to ANYONE who has calculations to solve.

The theory of the SLIDE RULE is quite simple and with a little practice proficiency in its operation may rapidly be developed. A knowledge of the few principles which underlie the workings of the Slide Rule will reduce the time required to learn its use as well as give you a feeling of security in the operation—a feeling that makes you know you are doing the right operation for the information you want to obtain.

The beginner should have no difficulty in mastering the use of the Slide Rule if he will study the instructions carefully and practice the various exercises given. GO SLOWLY AND SURELY, and much time will be saved and your errors will be few.

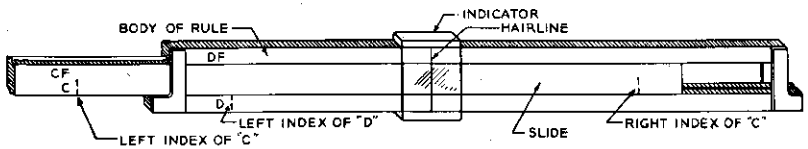


Fig. 1

2. General Description.

The slide rule consists of three main parts, see Figure 1; the BODY which is the fixed part of the rule, the SLIDE which can be moved left or right between the BODY, and the INDICATOR which slides either left or right on the face of the BODY and SLIDE. The INDICATOR has a fine hairline etched on the glass which is used for accuracy of settings and for marking results.

The mark on the scales associated with the primary number "1" is called the *index* of the scale. An examination of the "C" and "D" scales shows that they have two indices—one at the left end and one at the right end of the scale. Later when the "A" and "B" scales are used you will find three indices on each—left, middle, and right.

The Slide Rule has many scales which are for various operations. These scales will be identified as they are used. Pay attention only to those scales used in the particular instruction being given—all scales are explained. Study them one at a time as given in the following.

3. Theory of the Slide Rule.

In MULTIPLICATION by logarithms one adds the logarithms of the numbers to be multiplied. This sum is the logarithm of the answer. For instance, to multiply 2 x 4 by logarithms do the following:

$$\text{Log } 2 + \text{Log } 4 = \text{Log } 8 \qquad \begin{array}{r} \text{Log } 2 = 0.301 \\ + \text{Log } 4 = \underline{0.602} \\ \text{Log } 8 = 0.903 \end{array}$$

To DIVIDE by logarithms one subtracts the logarithm of the devisor of the numbers from the logarithm of the dividend. Therefore, to divide 4 by 2 using logarithms—do the following:

$$\text{Log } 4 - \text{Log } 2 = \text{Log } 2 \qquad \begin{array}{r} \text{Log } 4 = 0.602 \\ - \text{Log } 2 = \underline{0.301} \\ \text{Log } 2 = 0.301 \end{array}$$

The Slide Rule does these operations for you MECHANICALLY. Thus, you are saved the time and labor of looking up the logarithms of numbers in a table of logarithms and then subtracting or adding these logarithms.

4. Construction of the "C" and "D" Scales.

The logarithms of numbers from 1 to 10 are

Log 1 = 0.0000	Log 6 = 0.7782
Log 2 = 0.3010	Log 7 = 0.8451
Log 3 = 0.4771	Log 8 = 0.9031
Log 4 = 0.6021	Log 9 = 0.9542
Log 5 = 0.6990	Log 10 = 1.0000

These logarithms are to the base "10" and may be found in any table of logarithms.

Take these logarithms and a ruler or scale that is divided into inches and tenths of inches and actually lay the numerical values off as indicated in Figure 2a. Here a standard scale divided into tenths is shown and marks are drawn opposite the value of the logarithm —*THE NUMBER OPPOSITE THE LOGARITHM BEING THE NUMBER REPRESENTED BY THE LOGARITHM.*

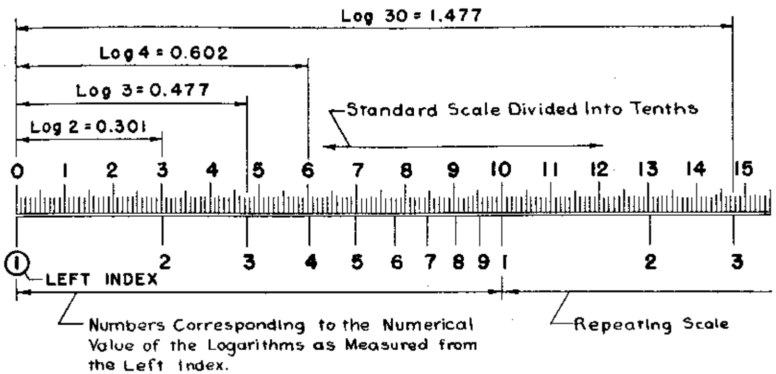


Fig. 2a

The distance from the "0" to 4.771 on the scale represents the logarithm of 3 which is 0.4771. Instead of calling this 0.4771 the number "3" is shown. Thus, only the numbers you are dealing with appear on any scale of the slide rule.

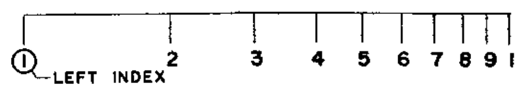


Fig. 2b

After constructing this you have a crude slide rule scale as shown in Figure 2b. Every 10 inches* the scale repeats itself—see Figure 2a.

**The so called 10-inch slide rule is actually 25 centimeters in length and not 10 inches.*

Figure 3 shows the "C" scale. This is just the "plot" of the logarithms of numbers from 1 to 10.

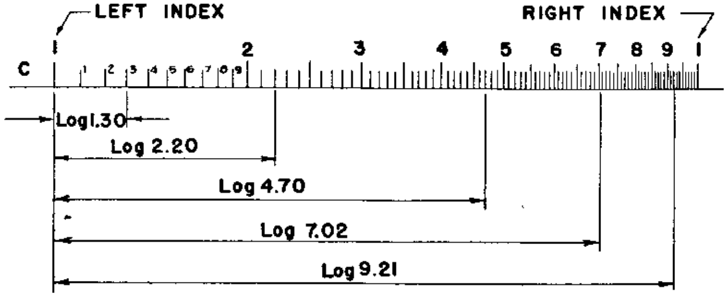


Fig. 3

For numbers above ten or below one the decimal part of the logarithm does not change—only the number to the LEFT of the decimal. The number to the left of the decimal is called the "*Characteristic*" while the numbers to the right are called the "Mantissa" of the logarithm.

THEREFORE, the "C" scale in Figure 3 is a "plot" of the "Mantissas" *of all numbers*—the "characteristic" not shown. The "characteristic" can be obtained by inspection—being defined as follows:

> For numbers greater than "1", it is *positive* and is one less than the number of digits to the *left* of the decimal point.

> For numbers less than "1", it is *negative* and is numerically one greater than the number of zeros *immediately following* the decimal point.

Thus, the logarithm of 2 is 0.3010, while the logarithm of 20 is 1.3010—or the logarithm of 200 is 2.3010. Since in 200 there are 3 digits to the LEFT of the decimal—the characteristic is $3 - 1 = 2$. The characteristic of the logarithm of 0.002 is -3. Since in 0.002 there are two zeros IMMEDIATELY FOLLOWING the decimal point, the characteristic is $-(2+1) = -3$, but the mantissa is still $+0.3010$. Therefore, the actual logarithm is $(-3 + 0.3010)$ which equals -2.6990. This last figure is not in the most convenient form with which to work so (for convenience) we write it as $(+7.3010 - 10)$. Since the $(+7 - 10)$ is -3 we still have the actual logarithm. Thus when the number is less than one its characteristic is written as indicated here—as *Log 0.002 =* $+7.3010 - 10$.

In Figure 3 the distance from the left "1" (called the LEFT INDEX) to the number 4.70 represents the mantissa of 4.70, 0.470, 470, or 4700 or any decimal multiple of 4.70—the proper characteristic being used in each case.

5. How to Read the Scale.

First, notice that the scale constructed in Article 4 is divided into numbers from 1 to 10—the right 1 (RIGHT INDEX) can be read as 10. Each space is divided into ten parts. These divisions are therefore approximately 1/10th of the space between the large division numbers. These subdivisions are further divided into decimal parts.

There are three sections of the scale where the subdivisions are different—between prime numbers 1 and 2; 2 and 4; and 4 to 10.

The number 14 would be located at the fourth long mark (4th tenth mark) after the prime number "1" (left index). The first short mark after the number 14 would be 141—the second short mark would be 142—the third short mark would be 143, etc. Therefore, each short mark between the first subdivisions represents the third digit of the number.

If the hairline of the indicator is placed as shown below, the reading would be 143.

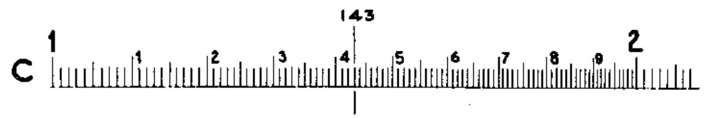


Fig. 4

The fourth digit of a number must be located by *estimation*. Thus, the number 1567 is estimated as

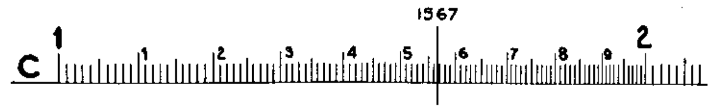


Fig. 5

In locating the fourth digit of any number falling between the prime numbers 1 and 2, the interval between the small divisions can be imagined divided into ten parts and the fourth digit estimated.

From the above, it appears that we may read four figures of a result in this section of the scale. This means an attainable accuracy of, roughly, 1 part in 1000, or one tenth of one percent.

In the second section of the scale, between the prime numbers 2 and 4 (Figure 6), the first subdivision marks (tenths) are *not* numbered. However, the halfway marks (five tenths); namely, 2.5 and 3.5, are, for convenience, longer than the other tenths' marks. There are ten subdivisions between the prime numbers and each of the subdivisions are divided into five parts—each part being 2/10ths of the first subdivision, or 2/100ths of the main division.

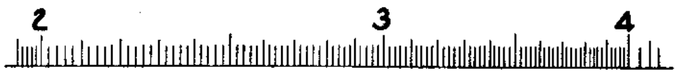


Fig. 6

The number 23 would be located at the third long subdivision mark after the prime number 2. The first short mark after the number 23 would be 232—the second short mark would be 234—the third short mark would be 236, etc. Therefore, each short mark between the first subdivisions represents the third digit of the numbers—and only the *even* ones. To obtain the location of the number when the third digit is an *odd* number, as 235, estimate halfway between the short divisions.

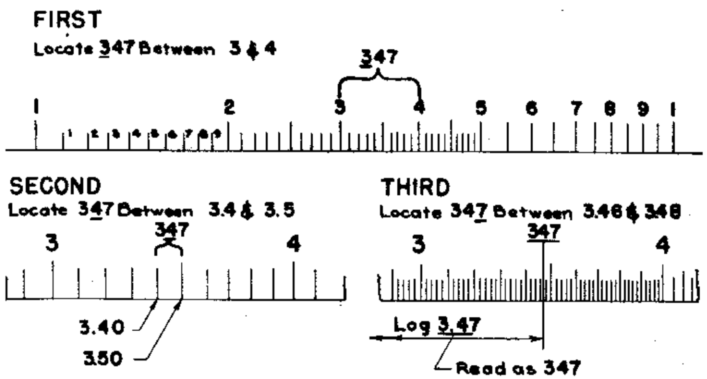


Fig. 7

To locate the number 347, determine the first digit—3 in this case. This indicates the number is between the prime numbers 3 and 4. Therefore, FIRST bring indicator hairline to prime number 3.

SECOND, locate the second digit—4—by bringing the indicator hairline to the fourth long subdivision mark following the prime number 3.

THIRD, locate the third digit—7—by moving the hairline halfway between the third and fourth short mark following this, 346 and 348, which therefore gives you the number 347 as shown in Figure 7.

In locating the fourth digit of any number falling between the prime numbers 2 and 4, the interval between the small divisions (two tenths can be imagined divided in half, and each of these halves (one tenth each) imagined divided into ten parts, and the fourth digit estimated.

In the third section of the scale, between the prime numbers 4 and 10, the first subdivisions (tenths) are not numbered, see Figure 8. However, the halfway marks (five tenths); namely, 4.5, 5.5, 6.5, etc., are, for convenience, longer than the other tenths' marks. There are ten subdivisions between the prime numbers and each of the subdivisions are divided into two parts—each part being 5/10ths of the first subdivision, or 5/100ths of the main division.

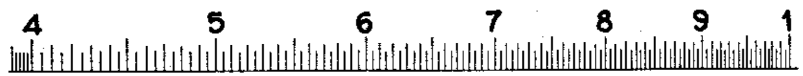


Fig. 8

The number 67 would be located at the seventh long subdivision mark after the prime number 6. The short mark after this would be 675.

To obtain the location of a third digit of any number falling between prime numbers 4 and 10, the interval between the first subdivisions (tenths) can be imagined divided into ten parts (the fifth part being already indicated by a short line), and the third digit estimated.

When there are more digits in a number than can be accurately read, "round off" the number to either four digits (if between prime numbers 1 and 4 on the scale), or three digits (if between prime numbers 4 and 10 on the scale). The number 12346 should be "rounded off" as 12350; the number 56783 should be "rounded off" as 56800.

Exercises

MAKE ALL SETTINGS ON YOUR SLIDE RULE

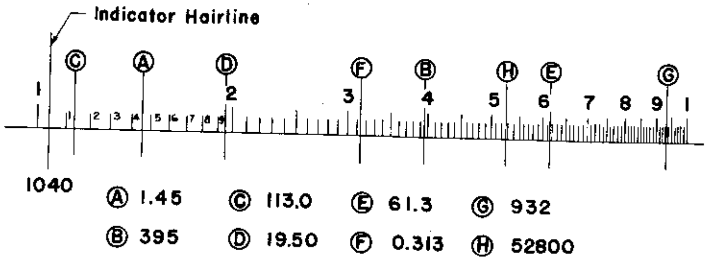
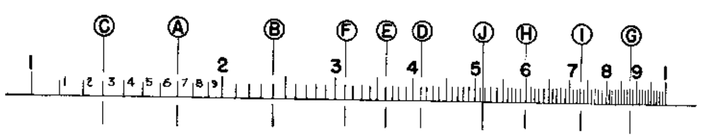


Fig. 9

In Figure 9, the hairline is first placed at 1040. Place the indicator hairline of your slide rule to this and the other placed as shown. Do you read the values given?

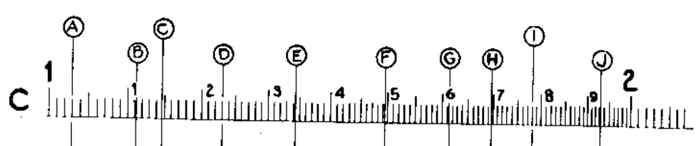
EXERCISE 1.



1. Record the Readings for the Hairlines Indicated.

A.______ C.______ E.______ G.______ I.______
B.______ D.______ F.______ H.______ J.______

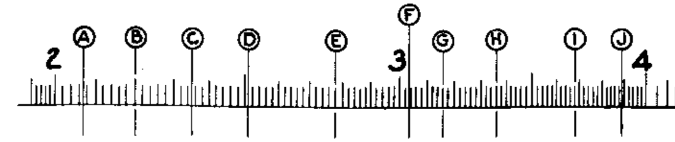
EXERCISE 2.



2. Record the Letter Opposite the Corresponding Number.

1230______ 1.612______ 0.01112______ 1.78______ 1696______
14.94______ 1.030______ 193.2______ 13.42______ 1.147______

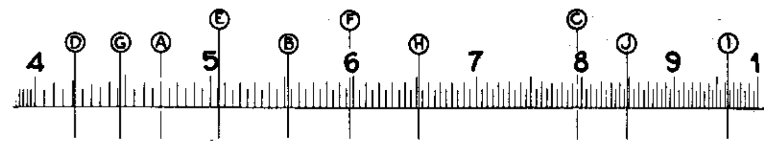
EXERCISE 3.



3. Record the Letter Opposite the Corresponding Number.

2.78______ 2350______ 2.51______ 0.368______ 2.07______
3.035______ 2.20______ 38.9______ 31.6______ 336______

EXERCISE 4.



4. Record the Scale Reading Opposite the Hairline Indicated.

A.______ C.______ E.______ G.______ I.______
B.______ D.______ F.______ H.______ J.______

ANSWERS TO EXERCISES ABOVE

1.	A.	17	C.	13	E.	36	G.	88	I.	73
	B.	24	D.	41	F.	31	H.	6	J.	515

2.	1230	D	1.612	G	0.01112	B	1.78	I	1696	H
	1494	F	1.030	A	193.2	J	13.42	E	1.147	C
3.	2.78	E	2350	C	2.51	D	0.368	I	2.07	A
	3.035	F	2.20	B	38.9	J	31.6	G	336	H

4.	A.	470	C.	795	E.	506	G.	447	I.	963
	B.	553	D.	421	F.	597	H.	652	J.	848

CHAPTER II

MULTIPLICATION AND DIVISION

6. Multiplication.

In the discussion on the theory of the Slide Rule it was stated that in order to multiply by the use of logarithms one added the logarithms of the numbers you intended to multiply.

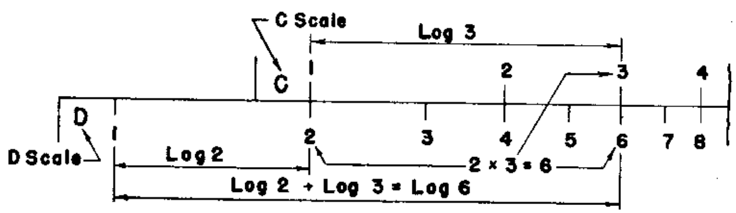


Fig. 10

In Figure 10 is indicated the multiplication of 2 × 3. Set the left index of the "C" scale opposite the "2" on the "D" scale. Move the INDICATOR to "3" on the "C" scale and opposite this on the "D" scale read the answer as **6**.

Note what you have actually done. In Figure 10 a distance equal to the logarithm of 2 has been added to a distance equal to the logarithm of 3. The sum of these logarithms is the logarithm of 6 which is the answer. Since the logarithms are not shown but only the numbers they represent, one can read the answer directly.

In Figure 11 the mechanical operation for multiplying 19.30 × 5 is indicated. What is actually done is the addition of the logarithm of 19.30 to the logarithm of 5 which gives you the logarithm of 96.5. This is read directly on the "D" scale.

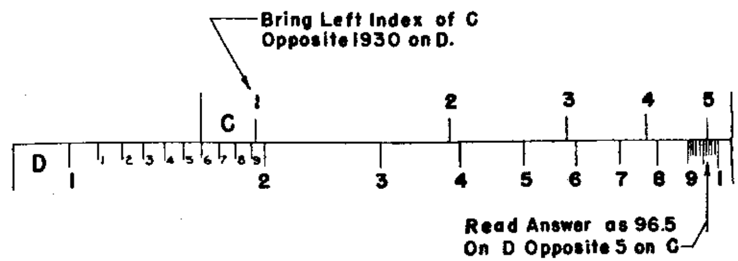


Fig. 11

The mechanical operation is performed as follows: First, set the left index of "C" opposite the number 1930 on "D"; second, move the indicator until it is at 5 on the "C" scale; and third, read the answer **965** *on the "D" scale opposite the 5 on the "C" scale. This does not give you the decimal point. However, 19.30 is about 20 and 5 × 20 is 100 Therefore, the answer is approximately 100. It is obvious then that the answer must be* **96.5**.

7. Accuracy of the Slide Rule.

The "C" and "D" scales of the "10-inch" Slide Rule can be read to three significant figures throughout their length. Between the left index and the primary number "2" one can estimate quite accurately to four significant figures. It is recommended that one not attempt to estimate beyond the third significant figure if the answer is to the right of the primary number "2" and beyond four significant figures when the setting is between the left index and the primary number "2". However, it is possible to make a rough estimate of the fourth significant figure between "2" and "4" but this last place should not be considered as too accurate.

8. Decimal Point.

The decimal point is best obtained by a quick mental calculation. For instance in Figure 12, the number 0.215 is multiplied by 0.229. The Slide Rule gives the answer as 492 which would be the same as if you multiplied any of the other decimal combinations as indicated in the Figure. Obviously the answer for all the possible decimal arrangements could not be 492. Therefore, the decimal point must be located.

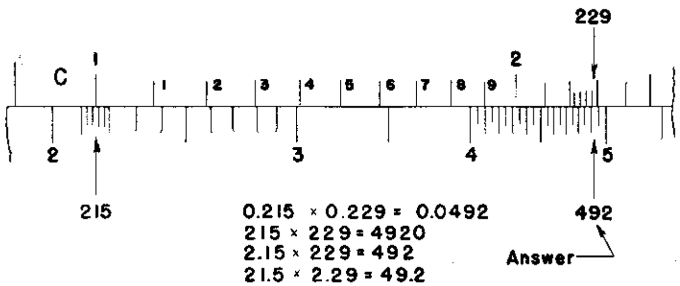


Fig. 12

Since 0.215 is approximately 0.2 and 0.229 is approximately 0.23, you can make the quick mental calculation of 0.2 × 0.23 = 0.046. This indicates you should read the slide rule answer as 0.0492.

Likewise, in more complicated problems you can locate the decimal by quick calculations. If you had the following to evaluate $\frac{145 \times 205 \times 68}{89 \times 12.5}$, you might write this as $\frac{150 \times 200 \times 70}{100 \times 10}$. A quick mental calculation of this would give 2,100,000 ÷ 1000 or 2100. Your answer would be approximately 2100 or four figures to the left of the decimal. The actual answer in this case is 1817. Such quick mental calculations are quite simple and the decimal point can be located accurately by this means.

9. Use of the Right Index in Multiplication.

In using the "C" and "D" scales to multiply numbers, such as 8 × 5—where one or both of the numbers are on the right end of the scales, the right index can be used.

*In Figure 13 is indicated the multiplication of 8 × 5. Set the right index of the "C" over 8 on the "D" scale. Move the indicator to 5 on the "C" scale and under the hairline read the answer—***40***—on the "D" scale.*

NOTE: If you had used the left index of "C" over 8 on the "D" scale, the answer which is read under the 5 on the "C" scale would have been off the rule.

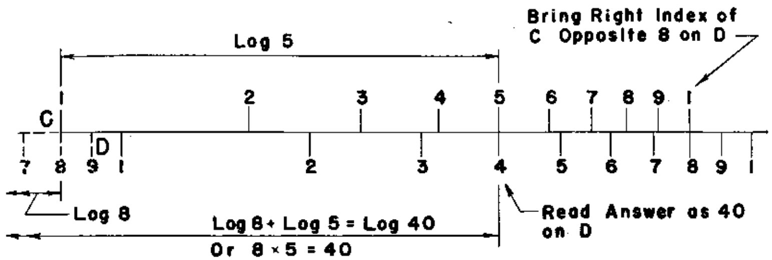


Fig. 13

Therefore, the right index and the left index of any of the scales can be used interchangeably, whichever will place the answer on the rule.

The reason for the above statement is that the "C" and "D" scales can be thought of as being continuous—or that they repeat themselves. In Figure 13 to the left of the LEFT INDEX of "D" is shown in "dotted" the numbers 7, 8, and 9. These are the same numbers and are placed identically as those on the right end of the actual "D" scale. Therefore, you can think of an imaginary scale to the left of the LEFT INDEX of the "D" scale.

In Figure 13, the right index of "C" is brought to 8 on "D". Notice that when this is done the left index of "C" is at 8 on the imaginary or "dotted" portion of "D". Now, the multiplication can be made as with any other numbers using the LEFT INDEX of "C". The answer is on "D" opposite 5 on "C".

Exercises

Fill in the following table, first, with the slide rule reading, and second, with the decimal point located correctly.

Exercise No.	Multiplication	Answer as Read on Slide Rule	Corrected Answer
1	2.45 × 31	760	76.0
2	345 × 3.46	______	______
3	972 × 0.45	______	______
4	1.035 × 0.081	______	______
5	23.1 × 1.03	______	______
6	758 × 123.46	______	______
7	4051 × 7.854	______	______
8	45.78 × 32.98	______	______
9	2.3 × 0.119	______	______
10	3.7 × 6.75	______	______

Multiply the following:

11. 3.5 × 798
12. 3.891 × 9243
13. 1.067 × 2.346
14. 78.9 × 2.453
15. 6.57 × 8.77
16. 45.03 × 77.7
17. 2.1 × 72.3
18. 0.00891 × 0.246
19. 0.0452 × 10089
20. 1.9099 × 103.4

ANSWERS TO EXERCISES ABOVE

Exercise No.	Answer	Exercise No.	Answer
2	1194	11	2790
3	437	12	36000
4	0.0838	13	2.50
5	23.8	14	193.5
6	93600.	15	57.6
7	31800.	16	3500.
8	1510	17	151.8
9	0.274	18	0.00219
10	25.0	19	456
		20	197.5

10. Division.

In dividing by logarithms one subtracts the logarithm of the divisor from the logarithm of the dividend in order to obtain the logarithm of the quotient or answer. This can be done by simple mechanical manipulation on the slide rule.

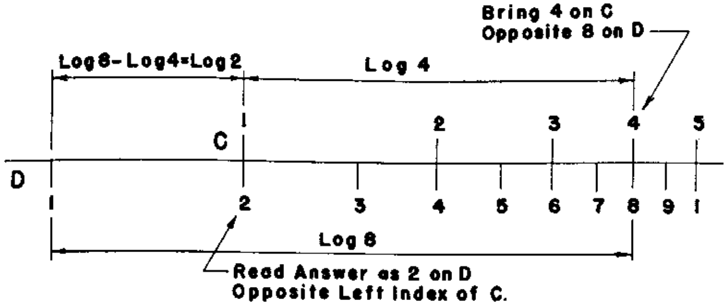


Fig. 14

In Figure 14 is indicated the division of 8 by 4. This is performed mechanically on the slide rule by the subtraction of the logarithm of 4 from the logarithm of 8.

Set the indicator at 8 on the "D" scale. Bring 4 on the "C" scale over 8 on the "D" scale and read the answer opposite the left index *of the "C" scale as 2 on the "D" scale.*

Note what you have actually done. In Figure 14 a distance equal to the logarithm of 8 (dividend) is located on the "D" scale, from which is subtracted a distance equal to the logarithm of 4 (divisor) on the "C" scale, leaving a distance equal to the logarithm of 2 (the quotient, or answer) on the "D" scale.

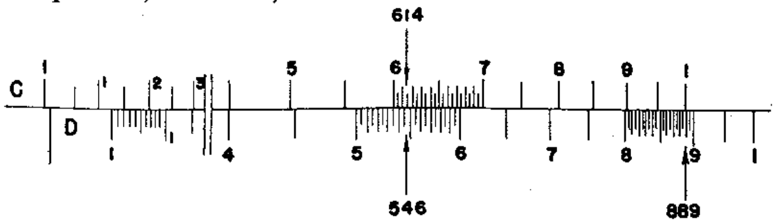


Fig. 15

The division of 546 by 614 is indicated in Figure 15. First, bring the indicator to the dividend, 546, on "D" and, second, bring to the hairline of the indicator 614 on the "C". You can then read your answer as 889 *on "D" opposite the right index on "C". The left index would also be opposite the answer but no scale of "D" exists at this point.*

You will find that if both the "C" and "D" scales repeated themselves an infinite number of times, the numbers opposite the indices of one would be the same. That is, if in Figure 15 the "C" and "D" scales were continuous and repeated themselves, the number opposite the indices on the "C" scale for this setting would always be 889 on "D" and that opposite the indices on the "D" scale would always be 1125 on "C".

In determining the location of the decimal when dividing 546 by 614, you can again make a quick mental calculation as 500 divided by 600 gives an answer a little less than 1. Therefore, the answer should read as 0.889. Likewise, if the number 546 is to be divided by 6.14, you can make the quick mental calculation as 540 divided by 6 gives 90. Your answer would then be 88.9 in this second case.

Fill in the following table, first, with the slide rule reading, and second, with the decimal point located correctly.

Exercise

Exercise No.	Division to be made	Answer as Read on Slide Rule	Corrected Decimal Point
1.	9.866 ÷ 2	493	4.93
2.	10.34 ÷ 31.4	______	______
3.	44.56 ÷ 1.239	______	______
4.	33.78 ÷ 98.7	______	______
5.	1245 ÷ 1.23	______	______
6.	3.46 ÷ 6.25	______	______
7.	3.3378 ÷ 22.89	______	______
8.	0.00033 ÷ 36.7	______	______
9.	0.0376 ÷ 0.0057	______	______
10.	1.34573 ÷ 6.784	______	______

Divide the following:

11. 87.5 ÷ 35.9	14. 0.0566 ÷ 5.47	17. 0.0348 ÷ 7.43
12. 3.45 ÷ 0.032	15. 3.42 ÷ 3.27	18. 3.142 ÷ 78.0
13. 1025 ÷ 9.71	16. 286 ÷ 3.45	19. 8.96 ÷ 44.5
		20. 1.773 ÷ 0.667

Answers to the above exercises:

2. 0.329	8. 0.00000899	14. 0.01035
3. 36.0	9. 6.60	15. 1.046
4. 0.342	10. 0.1984	16. 82.9
5. 1012	11. 2.44	17. 0.00468
6. 0.554	12. 107.8	18. 0.0403
7. 0.1458	13. 105.6	19. 0.201
		20. 2.66

11. Use of Reciprocals in Division.

The method of dividing 9 by 2 as explained above would be to bring the 2 on "C" opposite the 9 on "D". This could be done, but it requires that you bring the slide over to the right until it is almost out of the body of the rule. This division can be done in an easier manner by using the reciprocal of one of the numbers.

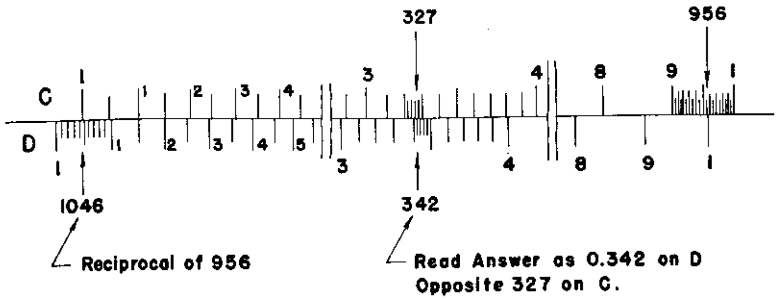


Fig. 16

In figure 16, the number 327 is divided by 956. This division is the same as if you multiplied 327 by $\frac{1}{956}$. Therefore, divide 1 by 956 first and then multiply the answer you obtain by 327.

In the figure, the 956 on "C" is brought opposite the right index on "D". The reciprocal or $\frac{1}{956}$ is read on "D" opposite the left index on "C". Move the indicator until it is at 327 on "C". Opposite this, read the answer as 342 on "D". Determine the decimal by mentally dividing 300 by 1000 giving 0.3. Therefore, the correct answer is 0.342. This manipulation saves the large movement of the slide. Now try the regular method of dividing 327 by 956. You will obtain the same answer. Next, try again the method just explained and notice the saving in time and labor.

AGAIN, THE SLIDE RULE IS A TOOL. Only when you are fully familiar with what it can do for you, can you reduce the amount of effort required in your calculations.

When dividing 277 by 11.24 as in Figure 17, you can use the same scheme as above. First divide 1 by 11.24. This is done by bringing the 1124 opposite the left index on "D". The reciprocal could be read at the right index of "C" on "D" but instead move the indicator to 277 on "C". Opposite this, read 246 on the" D" scale, which is 277 ÷ 11.24.

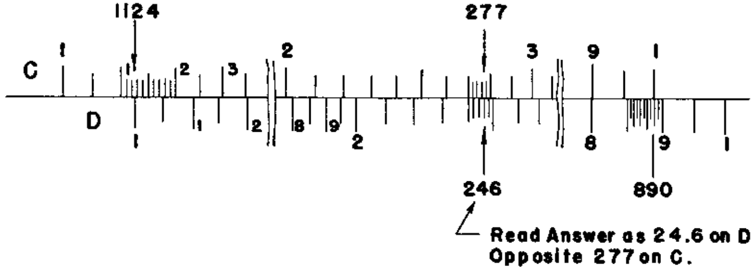


Fig. 17

12. The Folded Scales—"CF" and "DF".

The folded scales "CF" and "DF", in reality, are "C" and "D" scales cut in half at π(=3.1416) and the two halves switched, bringing the left and right indices to the middle as one index and π to each end in alignment with the left and right indices of the "C" and "D" scales. The "CF" and "DF" scales and their location with respect to the "C" and "D" scales are shown in Figure 18.

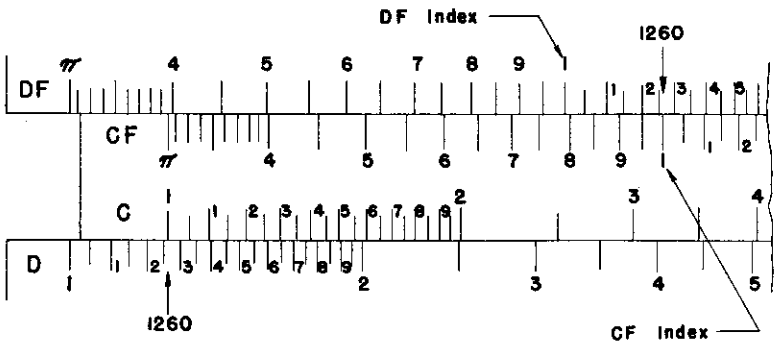


Fig. 18

This arrangement of scales has two distinct advantages:—Any number on the "C" and "D" scales is easily multiplied by π directly

above on the "CF" and "DF" scales. Thus, to multiply any number by π, bring the indicator hairline to the number on "D" and read the answer under the hairline on the "DF" scale. Likewise, one can divide a number by π by bringing the hairline to the number on the "DF" scale and reading the answer under the hairline on the "D" scale. For instance, to multiply π (=3.1416) by 2, bring the indicator hairline to 2 on "D", and above on "DF" read the answer —6.28. To divide 9.42 by π, bring the hairline of the indicator to 9.42 on "DF" and below read the answer 3 on "D".

The other advantage of the folded scales enables factors to be read without resetting the slide, which factors might be beyond the end of the rule when using only the "C" and "D" scales. *In effect, the use of the "CF" and "DF" folded scales is like extending a half scale length to each end of the "C" and "D" scales.*

Looking again to the "DF" and "CF" scales in Figure 18, you will notice that no matter what number the left index of "C" is placed opposite on "D", the middle index (the only index) of "CF" is opposite the same number on "DF". Likewise, wherever the "D" indices are with respect to the "C" scale, the "DF" index is opposite the same number on the "CF" scale. In Figure 18, the left index of "C" is opposite 1260 on "D". Also, the index of "CF" is opposite 1260 on "DF". Set your slide rule as in Figure 18. Notice the numbers opposite the right index of "D" and the index of "DF". These should be the same.

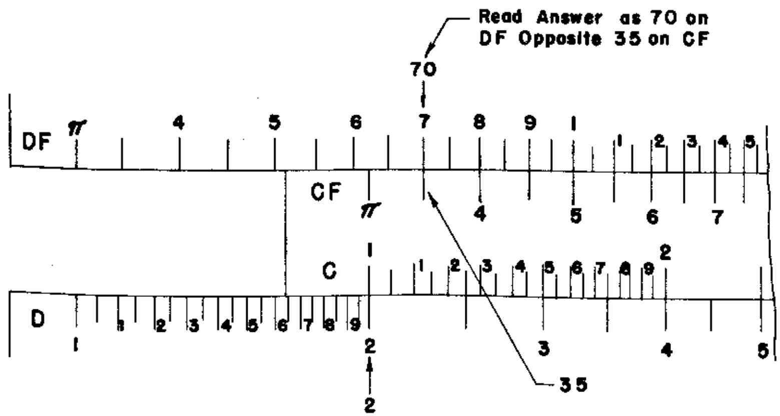


Fig. 19

In order to multiply by using the "CF" and "DF" scales, see the illustrated problem in Figure 19. Here 2 is multiplied by 35. *First set the left index of "C" opposite 2 on "D". The answer can be read on "D" opposite 35 on "C" as in the regular manner. Also, you can read the answer on "DF" opposite the 35 on "CF". Again, the answer is* **70.**

Using this same figure multiply 2×9. *The left index of "C" is placed opposite 2 on "D". Since 9 on "C" is off the scale on "D" you must read the answer as* **18** *on "DF" opposite 9 on "CF". This permits you to obtain the answer when it would otherwise be off the regular "D" scale.*

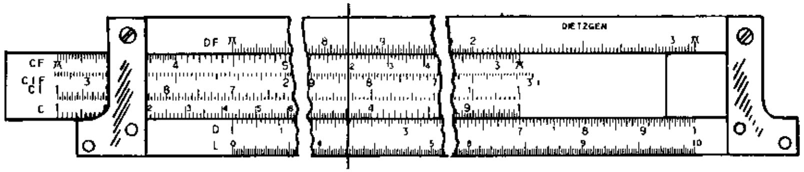


Fig. 20

To multiply 7×12 place the right index of "C" opposite 7 on "D", see Figure 20. Opposite the 12 on "CF" read **84** *on "DF", which is the answer.*

13. The Reciprocal Scales—"CI", "DI" and "CIF".

The reciprocal of a number is 1 divided by the number. Thus, the reciprocal of 8 is $\frac{1}{8}$ or 0.125.

The "CI", "DI" and "CIF" scales are reciprocal scales. They are constructed in a similar manner as the "C" ,"D" and "CF" scales—except, they are subdivided in the opposite direction. The "CI" and "DI" (or inverted "C" and "D" scales start with "1" at the right end and are subdivided from 1 to 10 from right to left. The numbers on the inverted "CI", "DI" and "CIF" scales are in red to help identify them and to make it easier for one to be sure the correct scale is being used.

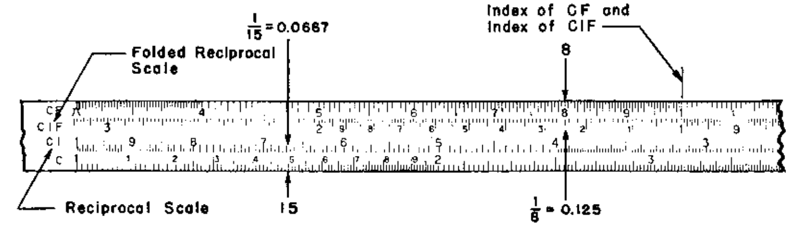


Fig. 21

Since the scales are inverted the indicator hairline can be brought to any number on the "C" scale and the reciprocal of that number can be read on the "CI" scale. Likewise, the reciprocal of any number shown on the "D" and "CF" scales can be found directly opposite the number on the "DI" or "CIF" scale.

For instance, determine the reciprocal of 15. *Bring the indicator hairline to 15 on the "C" scale and read directly above on the "CI" scale* **0.0667**, *see Figure 21. Also, for the reciprocal of 8, bring the hairline to 8 on the "CF" scale and directly below this read* **0.125** *on the "CIF" scale.*

These scales are used in connection with the "C", "D", "CF", and "DF" scales in multiplication and division. To multiply $12 \times 2 \times 15$, one must add the logarithms of the three numbers together. This sum will be the logarithm of the answer. To do this on your slide rule, the indicator is first brought to 12 on the "D" scale. Second, the slide is moved until the "2" on the "CIF" is at the indicator. Third, the indicator is moved to the 15 on the "CF" scale. Fourth, read the answer as **360** on "D" directly under the hairline.

What has actually been done? A distance equal to the logarithm of 12 has been added to a distance equal to the logarithm of 2, which would bring you to the index on "CIF". Since the index on the "CF" scale is at the same point, you can then add a distance equal to the logarithm of 15 by sliding the indicator to the right until it is at 15 on the "CF" scale. The answer is then read on the "D" scale under the hairline of the indicator.

Using the same setting of your rule multiply 12×2 and divide the result by 6. Set the indicator at 12 on the "D" scale as before. To this, bring the "2" on the "CIF" scale. Slide the indicator to 6 on the "CIF" scale and read the answer **4** on the "D" scale under the hairline.

Note what you have actually done is to multiply 12×2 by adding a distance equal to the logarithm of 12 to a distance equal to the logarithm of 2 giving you a distance equal to the logarithm of 24 (the product of 12×2) on the "D" scale. From the distance equal to the logarithm of 24 is subtracted a distance equal to the logarithm of 6. This last step ordinarily would be done by moving the indicator to the *left* from the index of "CIF". However, since our answer would be off the rule on the left and since we are dealing with folded scales, where the right section of the "CIF" scale is a continuation of the left section of the "CIF" scale, we can effect this subtraction mechanically by moving the indicator to the *right* to 6 on the "CIF" scale. Under the hairline at this point read the answer as 4 on the "D" scale.

The "DI" scale is particularly useful in the following types of numerical problems:

TYPE I. $\frac{X}{A}, \frac{Y}{A}, \frac{Z}{A}$ Where "A" is constant, and "X", "Y", and "Z" are variables. Suppose it is desired to find the individual quotients of a series of fractions; all having the *same denominator* but *different numerators*. Without the "DI" scale, it would be necessary to move the slide and indicator to solve for each fraction. However, with the "DI" scale, the slide need only be set one time and successive quotients can be obtained by merely moving the indicator.

EXAMPLE: Evaluate the following series of fractions: $\frac{14}{5}, \frac{15}{5}, \frac{16}{5}$.

Set the Right Index of "C" over the common denominator, *5*, on "DI".
Set the Indicator Hairline over *14* on "C".

Under the Hairline read the quotient of $\frac{14}{5}$, namely *2.8* on "D".

Next, move the Hairline to *15* on "C".

Under the Hairline read the quotient of $\frac{15}{5}$, namely *3* on "D".

Next move the Hairline to *16* on "C".

Under the Hairline read the quotient of $\frac{16}{5}$, namely *3.2* on "D".

TYPE II. $\frac{1}{X \times Y}$

Without the "DI" scale, the solution to a problem of this type would require 2 settings of the slide, and 2 settings of the Indicator. If the problem were one of conventional multiplication, that is, simply X × Y, the answer would be read on the "D" scale. However, since the reciprocal of the product is required, the answer may be read directly on the "DI" scale—which is the reciprocal of the "D" scale. By means of the "DI" scale, problems of this type can be solved directly with only one setting of the slide and one setting of the Indicator.

EXAMPLE: Evaluate $\frac{1}{2 \times 3}$

Set Left Index of "C" over *2* on "D".
Move Indicator Hairline to *3* on "C";
Under the Hairline, read .1666 on "DI".

EXERCISES

1. 3.45 ×54.7 ×106.8
2. 90.8 ×35 ÷55.8
3. 77.5 ×45.7 ÷(3.3 ×3.6)
4. 12.78 ×23.4 ÷301.5
5. 145 ×36.5 ×347.0 ÷(23.1 ×44.7)
6. 23.4 ×1.467 ×5.34 ÷(1.67 ×6.78)
7. 0.034 ÷(1.007 ×3.46)
8. 0.965 ×0.1045 ÷0.00884
9. 3.36 ÷(2.33 ×4.05)
10. 56.78 ÷(0.008 ×12.01)

Repeat the above calculations using not more than TWO settings of the slide. Use the "C", "D", "CF", "DF", "CI", "DI", and "CIF" scales in combination to help you solve the problems.

11. Find (a) 7.67 per cent of 19.45
    (b) 56.4 per cent of 356.9
    (c) 19.4 per cent of 524.8
    (d) 35.2 per cent of 1235.85

12. Evaluate (a) $\frac{14}{8.6}, \frac{68.5}{8.6}, \frac{142}{8.6}$
    (b) $\frac{14.3}{6.28}, \frac{65.9}{6.28}, \frac{84.6}{6.28}$ Use "CF" and "DF" Scales
    (c) $\frac{1}{1.6x4}$
    (d) $\frac{1}{3.14x.6}$

13. What per cent of
    (a) 57 is 18.3?
    (b) 33.6 is 30.4?
    (c) 78.4 is 89.6?
    (d) 445 is 65.8?

14. Using the "D" and "DF" scales, do the following:
    (a) 345.6 × π
    (b) 2.48 × π
    (c) 99.24 ÷ π
    (d) 44.5 ÷ π

ANSWERS to the Exercises above.

Exercise No.	Answer	Exercise No.	Answer	Exercise No.	Answer
1	20200	10	591	13 (a)	32.1%
2	56.9	11 (a)	1.492	(b)	90.5%
3	298	(b)	201	(c)	114.3%
4	0.992	(c)	101.8	(d)	14.79%
5	1779	(d)	435	14 (a)	1086
6	16.19	12 (a)	1.629, 7.96, 16.5	(b)	7.79
7	0.00976	(b)	2.28, 10.5, 13.5	(c)	31.6
8	11.41	(c)	.1562	(d)	14.16
9	0.356	(d)	.531		

CHAPTER III

PROPORTIONS

14. Proportion.

The ratio of two numbers X and Y is the quotient of X divided by Y written as $\frac{X}{Y}$. The ratio of 4 to 12 is $\frac{4}{12}$ or $\frac{1}{3}$. A *proportion* is an equation stating that two ratios are equal. Thus,

$$\frac{4}{12} = \frac{1}{3}; \quad \frac{X}{7} = \frac{3.7}{45.0}; \quad \text{or} \quad \frac{X}{Y} = \frac{C}{D}$$

are all proportions. These are often read as *"4 is to 12 as 1 is to 3"*, or *"X is to 7 as 3.7 is to 45.0"*, or again *"X is to Y as C is to D"*.

Many problems are solved by means of proportions. Generally only one of the four quantities is unknown as in the case with the second proportion above *"X is to 7 as 3.7 is to 45.0"*.

ILLUSTRATION:

If a 60 miles per hour speed is equivalent to 88 feet per second, how many feet per second is a car traveling that is moving at a speed of 75 miles per hour?

Set the proportion up as follows with "X" as the unknown: 88 feet per second is to 60 miles per hour as "X" is to 75

or $\frac{88}{60} = \frac{\text{"X"}}{75}$

The proportion can often be made as in the above illustration and the equation solved for "X".

15. Use of Proportion.

Proportions can easily be solved on the slide rule because of the following property of the "C" and "D" scales (also "A" and "B", as well as the "CF" and "DF" scales):

With the slide in any position, the ratio of any number on "C" to its opposite number on "D" is the same as the ratio of any other number on "C" to its opposite on "D".

This means that any two numbers on "C" together with their opposites on "D" form a proportion. Thus if 8 on "C" is set opposite

6 on "D", we also have 4 on "C" opposite 3 on "D", and 2 on "C" opposite 1.5 on "D". This is illustrated in Figure 22. The proportions can be read as "8 is to 6 as 4 is to 3 as 2 is to 1.5".

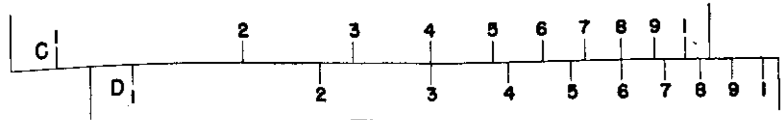


Fig. 22

The above is true of the "A" and "B" scales and the "CF" and "DF" scales as well. These additional scales will be found to be handy in the solution of proportions.

To illustrate the use of the rule for the solution of proportion problems, let us return to the illustration of article 14. Here we wanted to know the number of feet per second to which 75 miles per hour is equivalent. Write the proportion as

$$\frac{X}{75} = \frac{88}{60}$$

and make the following setting of your rule:

To 88 on "D" set 60 on "C"
Opposite 75 on "CF", read **110** *on "DF".*

In the above illustration, you have divided 88 by 60 and then multiplied this result by 75. You may always resort to straight multiplication and division for the solution of proportion problems if you wish.

It will be noted that in this illustration, the "CF" and "DF" scales were used. This is due to the fact that 75 on "C" is "off" the scale on "D"—thus we must use the folded scales.

ILLUSTRATION: A man receives $65.00 for a 40 hour week. How much should he receive for 33.5 hours of work at this rate?

Set the proportion up as *"X" is to 33.5 hours as $65.00 is to 40 hours.* Write this as

$$\frac{X}{33.5} = \frac{65}{40}$$

To 65 on "D", set 40 on "C"
Opposite 33.5 on "C" read the amount **$54.40** *on "D"*

Sometimes it may become necessary to solve a series of unknowns that can be set up as proportions. For example, let us determine the numerical values of the lettered quantities in the following proportion:

$$\frac{34.2}{4.65} = \frac{X}{34.5} = \frac{547.0}{Y} = \frac{2.312}{Z}$$

This proportion can be solved for the unknowns by one setting of the rule as follows:

To 34.2 on "D", set 4.65 on "C"

In this, note that the numerator of the ratio is on the "D" scale while the denominator is read on "C". Likewise, the numerators and denominators of the other ratios in the proportion will be read respectively on the "D" and "C" scales.

Thus, to obtain the unknowns with this setting of the rule
*Read X = **254** on "D" opposite 34.5 on "C"*
*Read Y = **74.4** on "C" opposite 547 on "D"*
*Read Z = **0.3145** on "C" opposite 2.312 on "D"*

The decimal point in each case was determined by approximating the known ratio. Thus, approximately 35 is to 5 as 7 is to 1. Therefore, in each case the ratio is a little less than 7 to 1.

ILLUSTRATION: If there are 16 ounces in 1 pound, how many ounces in 3.45 pounds?

First, set this up into a proportion that reads as follows:

"16" is to 1 as "X" is to 3.45.

To 16 on "D", set 1 on "C"
*Opposite 3.45 on "C" read **55.2** on "D"*

This illustrates the possibility of using the number "1" in a proportion. Often this is of considerable value in making proportion calculations.

Percentage problems can be solved quickly by the use of the proportion principle.

ILLUSTRATION: Find 37% of 1352.

37% of 1352 is the same as $\frac{37}{100}$ of 1352, or

0.37×1352

To 1352 on "D", set left index of "C"
*Opposite 0.37 on "C" read **500** on "D"*

Write the above in a proportion form.

$$\frac{37}{X} = \frac{100}{1352}$$

The setting is the same but in this form we can easily see that if one wanted any other definite percentage of the whole (1352), it could easily be obtained with this one setting.

ILLUSTRATION: A company's total sales in the four states of Illinois, Indiana, Michigan, and Ohio were $186,500. What percentage of the total sales were the sales in the respective states if these were: Illinois, $51,200; Indiana, $35,700; Michigan, $63,100; and Ohio $36,500.

Write the following proportion:

$$\frac{100}{186{,}500} = \frac{X}{51{,}200} = \frac{Y}{35{,}700} = \frac{W}{63{,}100} = \frac{Z}{36{,}500}$$

To 186500 on "D", set left index of "C"
Opposite the sales figures for the four states on "D", read the percentages on "C",

*You should read X = **27.46%**, Y = **19.14%**, W = **33.83%** and Z = **19.57%** respectively.*

To check these percentages—their sum must be 100% which is the case in this illustration.

Exercises

In each of the following exercises, determine the value of the unknown quantities. If the exercise is not set up in proportion form, set it up first in this form before solving the exercise.

1. $\frac{Y}{6.73} = \frac{81}{109}$

2. $Y = \frac{14 \times 0.787}{3.45}$

3. $\frac{X}{2.81} = \frac{3.92}{5.41} = \frac{4.32}{Z} = \frac{Y}{8.92}$

4. $\frac{17}{38} = \frac{X}{9} = \frac{10}{Z}$

5. $407 = \frac{71.2\ X}{48.3}$

6. $\frac{1}{4.28} = \frac{Z}{9.39}$

7. $\frac{33\ Z}{4.58} = 9.78$

8. $Y = \frac{8.71 \times 4.32}{3.21}$

9. $\frac{7.92}{84.32} = \frac{0.695\ X}{392.5}$

10. $4.81\ Y = \frac{0.281 \times 7.45}{3.81} = \frac{Z}{9.1}$

11. A head of a family receives $360 per month for his services and he uses this in the following manner: Clothes 15%, rent 25%, savings 12%, church 5%, recreation 5%, food 23%, car 5%, and misceloneous 10%. Determine the amount this head of the family spent an each item.

12. A distributing organization had the choice of four railroads to ship their merchandise on and they shipped in one year a total of 2,345,000 tons of merchandise. Railroad A received 540,000 tons, railroad B received 302,000 tons, railroad C received 756,000 tons, and railroad D received 747,000 tons. What percentage did each railroad receive?

Answers to the above exercises.

1. Y = 5.01
2. Y = 3.19
3. X = 2.04, Z = 5.96, Y = 6.46
4. X = 4.02, Z = 22.35
5. X = 277
6. Z = 2.19
7. Z = 1.356
8. Y = 11.73
9. X = 53.1
10. Y = 0.1142, Z = 5.00
11. 54, 90, 43.25, 18, 18, 82.75, 18, and 36 dollars
12. 23.05, 12.87, 32.24, and 31.84 per cent.

CHAPTER IV

SQUARES AND SQUARE ROOTS

Using "A" and "B" Scales

16. Squares.

In solving problems, there are many occasions when a number must be multiplied by itself. Thus, the area of a square 4 ft. on each side is 4 × 4 (or 4^2) which equals 16 sq. ft. This operation is called squaring.

Instead of writing 4 × 4, or 35 × 35, or any other number multiplied by itself, the operation is indicated by writing 4^2, or $(35)^2$. This is read 4 squared, or 35 squared—sometimes read as 4 or 35 to the second power.

You will find that it is always possible to square a number by using the "C" and "D" scales. A shorter method is to use the "A" and "D" scales, or the "B" and "C" scales.

THE "A" AND "B" SCALES, which are exactly alike, are what are called two-unit logarithmic scales; that is, two complete logarithmic scales which, when placed end to end, equal the length of the single logarithmic scale "D" or "C", in connection with which they are usually used. You will note by the fact that these two-unit logarithmic scales "A" and "B" are directly above the single-unit logarithmic scale "D" that when the hairline of the indicator is set to a number on the "D" scale, the square of the number is found directly above under the hairline on the "A" scale. Likewise, if the hairline is set to a number on the "C" scale, the square of that number is found under the hairline on the "B" scale.

Note that dual faced rules, having graduations on both sides, have an encircling indicator permitting any one of the scales on one side to be read in connection with any of the scales on the opposite side. Thus, if the hairline of the indicator is set to 2 on "C", the square of 2, namely, 4, will be found under the hairline on the opposite side of the rule on scale "B".

Note also that since the "A" and "B" scales are each two complete logarithmic scales, they can be used for multiplication and division the same as the "C" and "D" scales; as, for example, to multiply 2 × 3, set either the left or the middle index of "B" under either the 2 on the first unit of "A" or under 2 on the second unit of "A" and above 3 on "B", read the answer 6 on "A" in either the first or the second unit.

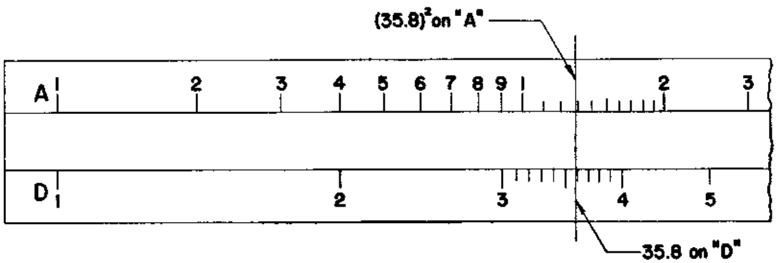


Fig. 23

ILLUSTRATION: What is the square of 35.8 or what is $(35.8)^2$? See Figure 23.
Set indicator at 35.8 on the "D" scale
Read answer on the "A" scale under the hairline as **1282.**
(The fourth digit being estimated)
Obtain the decimal by estimation as 40 × 40 is 1600
Therefore read the answer as **1282.**

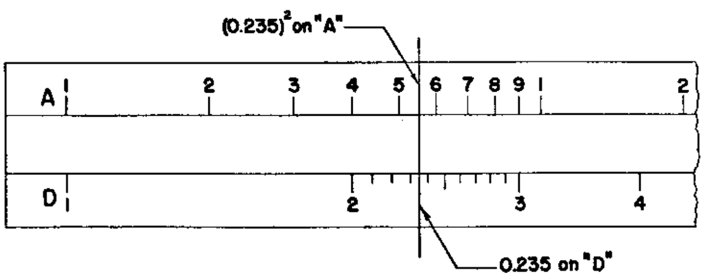


Fig. 24

ILLUSTRATION: What is 0.235 squared? See Figure 24.
Bring indicator to 235 on "D"
Read answer as **552** *on "A" under hairline*
Estimate decimal point by 0.2 × 0.2 which equals 0.04
Read answer as **0.0552.**

17. Applications of Squares.

The area of a circle is given as $\frac{\pi D^2}{4}$. This involves the square of the diameter.

Determine the area of a circle of 12″ diameter. Bring the indicator to 12 on "D", the square of which is 144 and is read immediately above under the hairline on "A". This is then multiplied by π by moving the left index of "B" under the hairline and sliding the indicator to π on "B", the product of which is immediately under the hairline on "A". Hold this product under the hairline on "A" and divide same by 4 which is done by moving 4 on the "B" scale under the hairline and reading the answer **113.0** on "A" immediately above the left index "B".

This operation could have been performed using the "A", "C", "D", and "DF" scales as follows: Use the "D" and "A" scales as above to obtain the square of 12. Since our calculation involves dividing by 4, we can effect this division by multiplying by the reciprocal of 4. Therefore, using the "C" and "D" scales, set 4 on "C" over the right index of "D" and read the reciprocal of 4 under the left index of "C". This reciprocal is then multiplied by $(12)^2$ or 144, by moving the indicator hairline to 144 on "C". This product can then be immediately multiplied by π by reading the answer **113.1** directly above under the hairline on "DF".

In this second method, you will be able to read to four significant figures (since you are between the prime numbers 1 and 2 of the rule), while in the first method, only three significant figures can be read on the "A" scale. It might be well that you do both of these operations again to familiarize yourself with the advantage of one method over the other.

In solving problems involving both multiplication and division, it is not necessary to read intermediate answers of each step in the calculation for all we are interested in is the final result. The best way to approach problems of this kind is to perform alternately—first, division; then multiplication; then division; then multiplication, and continue in this manner until the problem is solved. This minimizes the number of settings of the slide and the movement of the indicator.

ILLUSTRATION: Do the following indicated operation:

$$\left[\frac{45.8 \times 31.9}{5.6}\right]^2$$

Set the indicator at 45.8 on "D"
Bring 5.6 on "C" to hairline
Move indicator to 31.9 on "C"
Under hairline on "A" read **681**

Estimate decimal by 50 × 30 ÷ 5 equals 300
300 squared is 90,000
Therefore, answer should be **68,100**

The area of a circle was given above as $\frac{\pi}{4}$ times the diameter squared. π is 3.14 and $\frac{\pi}{4}$ is 0.785. Therefore, the area of a circle is the constant (0.785) times the diameter squared. Toward the right end of both the "A" and "B" scales is a long mark at 0.785 or $\pi/4$.

To obtain the area of a 12″ circle, bring the 0.785 mark on "B" to the right index of "A". Move the indicator to 12 on "D" and read the answer on "B" under the hairline as 113.0. In this operation, you are multiplying 0.785 by the square of 12. *Thus, to obtain the area of any circle, bring 0.785 mark on "B" to right index of "A". Bring indicator to the diameter on "D" and read answer on "B" under the hairline.*

Exercises

1. Use the slide rule to find the squares of each of the following numbers: 23, 33, 0.31, 87, 3358, 1.334, 6.78, 2.09, 31.9, 0.978, 31×10^3, 0.0065.

2. Determine the area of the circles (perform the operation in at least two ways) having the following diameters: (a) 3.45 ft., (b) 35 in., (c) 2.45, and (d) 12.5 in.

3. Do the following operations and square the answers:

a. $\frac{3.67 \times 7.34}{15.89}$ c. $\frac{0.89 \times 34.24}{1 + 34.1 \times 3.0}$

b. $\frac{67 + 4.5}{2.1 \times 34.5}$ d. $\frac{79.67 \times 3.45}{5.35}$

e. $\frac{3967 + 5280}{12300}$ f. $\frac{5.81 \times 9.89}{689.7}$

ANSWERS TO THE ABOVE EXERCISES.

1. 529, 1089, 0.0961, 7570, 11,270,000, 1.78, 46.0, 4.37, 1018, 0.956, 961×10^6, 42.3×10^{-6}.

2. (a) 9.34 sq. ft. (b) 962 sq. in. (c) 4.71 sq. ft. (d) 122.7 sq. in.

3. (a) 2.87 (d) 2640
(b) 0.974 (e) 0.566
(c) 0.874 (f) 0.00694

18. Square Roots.

The square root of any number is another number whose square is the first number. Five squared is 25 and the square root of 25 is 5. The symbol for the square root is $\sqrt{\quad}$. Thus to indicate the square root of 25 the symbol is used as $\sqrt{25}$.

ILLUSTRATION:

$\sqrt{9} = 3$ $\sqrt{100} = 10$
$\sqrt{16} = 4$ $\sqrt{121} = 11$
$\sqrt{49} = 7$ $\sqrt{169} = 13$

The square root of a number is found on the slide rule by reversing the process used in finding the square of a number; namely, locating the number whose square root is desired on the "A" scale and reading the square root of same under the indicator on the "D" scale.

The "A" scale has two parts that are identical. This scale is divided into divisions from 1 to 10 in one-half the length of the rule and again into divisions from 1 to 10 in the second half of the rule. The "B" scale is identical with the "A" scale. The first half of the "A" and "B" scales will be referred to as A-LEFT or B-LEFT and the other half as either A-RIGHT or B-RIGHT.

In order to find the square root of numbers with an *odd number of digits* to the left of the decimal point, use the A-LEFT scale in conjunction with the "D" scale.

ILLUSTRATION: What is the square root of 9 or 900?

Bring the indicator to 9 on A-LEFT
Under the hairline on "D" read the square root as **3.**

For 900 make the same setting
Read the answer as **30.**

To find the square root of any number with an *even number of digits* to the left of the decimal point, use the A-RIGHT scale in conjunction with the "D" scale.

ILLUSTRATION: What is the square root of 16 or 1600?

Bring the indicator to 16 on the A-RIGHT scale
Under the hairline, read **4** *on the "D" scale*

Or if the number is 1600, the setting is the same.
In this case, read the answer as **40.**

A study of the above two illustrations indicates that your answer should have one place to the left of the decimal for each two digits left of the decimal of the original number if it was an even number. When the original number is odd, add 1 to the number of digits and divide by 2.

To find the decimal point for the $\sqrt{7854}$, add the number of digits and divide by 2. Thus there are four digits and, therefore, the answer should have two digits to the left of the decimal.

To find the decimal point for $\sqrt{785.4}$, add 1 to the number of digits and divide by 2 again. Since there is an odd number of digits, add 1 giving 4 and divide by 2 giving 2 places to the left of the decimal point.

ILLUSTRATION: What is the square root of 7854?
The number has an even number of digits. Therefore:

Bring the indicator to 7850 on the A-RIGHT scale
Read the answer **88.6** *on "D" under the hairline.*

What is the square root of 785.4?
This number has an odd number of digits. Therefore:
Bring the indicator to 785.0 on the A-LEFT scale,
Read the answer as **28.0** *on "D" under the hairline.*

In both of the above cases the number was "rounded off" to three significant figures, to be within the accuracy of the rule.

19. Square Roots of Numbers Less Than Unity.

If the square root of a number less than unity is desired, move the decimal point to the RIGHT an even number of places until you have a number greater than 1. Thus to obtain $\sqrt{0.000347}$, change the number to read the $\sqrt{3.47}$. Obtain the $\sqrt{3.47}$ as before which is 1.864. Since the decimal point was moved 4 places to the right in the first operation, move it back to the left *half* of this amount. You would then read the answer as **0.01864.**

ILLUSTRATION: What is the square root of 0.0956?
Move the decimal 2 places to the right to obtain 9.56

Set the indicator at 9.56 on A-LEFT
Under the hairline read **3.09** *on "D".*

Finally, move the decimal $\frac{1}{2}$ *of 2 places to the left*

Read the answer as **0.309**

ILLUSTRATION: What is the square root of 0.0000158?
Move the decimal 6 places to the right to obtain 15.8

Set indicator at 15.8 on A-RIGHT
Under the hairline read **3.97**

Move the decimal $\frac{1}{2}$ *of 6 places to the left*

Finally the answer should be read as **0.00397**

Exercises

1. Find the square roots of each of the following numbers: 3, 30, 785, 78.5, 9.8, 98, 0.81, 0.081, 0.000152, 0.0000152, 35580, 1210.

2. The area of a circle is $\frac{\pi D^2}{4}$. If $\frac{\pi}{4}$ is 0.785, determine the diameter of the circles having the following areas: (a) 345 sq. ft., (b) 144 sq. in., (c) 0.724 sq. ft., (d) 192,000 sq. ft.

3. Determine the length of the sides of squares having the following areas: (a) 23.56 sq. ft., (b) 324.5 sq. in., (c) 3,458 sq. in., (d) 1.3786 sq. ft.

ANSWERS TO THE ABOVE EXERCISES.

1. 1.732, 5.48, 28.0, 8.86, 3.13, 9.90, 0.9, 0.285, 0.01233, 0.0039, 188.7, 34.8.
2. (a) 20.95 ft., (b) 13.54 in., (c) 0.96 ft., (d) 494 ft.
3. (a) 4.86 ft., (b) 18.0 in., (c) 58.8 in., (d) 1.175 ft.

20. Combined Operations Involving Squares and Square Roots.

The "B" and "C" scales can be used in the same manner as the "A" and "D" scales to obtain the square roots of numbers. This makes various combined operations easy with the slide rule.

For example, to obtain the result of $4 \times \sqrt{354}$, set the left index of "C" at 4 on "D" and move the indicator to 354 on "B-LEFT". Read the answer as **75.2** on "D" under the hairline. Likewise, to obtain the result of $8.6 \times \sqrt{34.8}$, set the right index of "C" at 8.6 on "D" and move indicator to 34.8 on "B-RIGHT". Read the answer as **50.7** on "D" under the hairline.

For simplicity the following general form will be used for all slide rule settings:

What is the value of $23.4\sqrt{7.86}$?

To 23.4 on "D", set 1 on "C"
Opposite 7.86 on B-LEFT read **65.6** *on "D"*

The same plan is used below for evaluating $94 \div \sqrt{34.9}$.

To 94 on "D", set 34.9 on B-RIGHT
Opposite 1 *on "C" read* **15.92** *on "D"*

To find the value of $\frac{8.78\sqrt{2.35}}{67.4}$ perform the operation as follows:

To 8.78 on "D", set 67.4 on "C"
Opposite 2.35 on B-LEFT read **0.200** *on "D"*

The reciprocal scale, "CI", can be used for evaluating $x = \frac{4.51}{21.2\sqrt{32.8}}$ as follows:

To 4.51 on "D", set 32.8 on B-RIGHT
Opposite 21.2 on "CI" read **0.0371** *on "D"*

SPECIAL EXAMPLES: Make all indicated operations with your own rule.

Example 1. Evaluate $\frac{0.356\sqrt{0.078} \times 54.3}{\sqrt{46.8}}$

To 0.356 on "D", set 46.8 on B-RIGHT
Move indicator to 0.078 on B-LEFT
Set 54.3 on "CI" to hairline
Read answer as **0.789** *on "D" opposite 1 on "C"*

Reviewing these operations you have done the following: First, 0.356 has been divided by $\sqrt{46.8}$ and, second, this result has been multiplied by $\sqrt{0.078}$. In the third step, you have multiplied by 54.3 by using your "CI" scale. Your slide rule in this third operation adds the logarithm of 54.3 to the logarithm of the result of step 2.

Example 2. Evaluate $\frac{\sqrt{89.5}\ (43.2)^2}{31.6 \times 903}\ (\pi)$

To 89.5 on A-RIGHT, set 903 on "C"
Move indicator to 31.6 on "CI"
Bring 43.2 on "CI" to hairline
Opposite 43.2 on "C" read **1.94** *on "DF"*

Reviewing these operations, you have done the following: First, $\sqrt{89.5}$ has been divided by 903 and, second, this result has been divided by 31.6. Third, the second result has been multiplied by 43.2 by using the "CI" scale and, fourth, this result has been again multiplied by 43.2 using the "C" scale. Finally, you have multiplied by π when the answer is read on the "DF" scale.

Example 3. Evaluate $\frac{\sqrt{4.35} \times \sqrt{54.2}\ (2.3)^2}{(8.4)^2\sqrt{34.9}}$

To 54.2 on A-RIGHT set 8.4 on "C"
Move indicator to 4.35 on B-LEFT
Bring 2.3 on "CIF" to hairline
Move indicator to 2.3 on "CF"
Bring 8.4 on "C" to hairline
Move indicator to left "C" index
Bring 34.9 on B-RIGHT to hairline
Read answer as **0.01947** *opposite right index of "D".*

Reviewing these operations, you have done the following: First, $\sqrt{54.2}$ has been divided by 8.4; second, this result has been multiplied by $\sqrt{4.35}$; third, the second result has been multiplied by 2.3 by using the "CIF" scale (dividing a product by the reciprocal of a number gives the same result as multiplying by the number); fourth, this result has been again multiplied by 2.3 using the "CF" scale; fifth, this result has again been divided by 8.4 using the "C" scale. Sixth, or finally, this result has been divided by $\sqrt{34.9}$ using the "B-RIGHT" scale and the answer **0.01947** is read on the "D" scale.

Exercises

In each of the following exercises perform the indicated operation.

1. $\sqrt{\dfrac{0.932}{0.012}}$
2. $\sqrt{\dfrac{3.26 \times 281}{0.821}}$
3. $\dfrac{3.83\sqrt{81.3}}{0.65}$
4. $\dfrac{(3.18)^2\ (\pi)}{\sqrt{3.91}}$
5. $\dfrac{37.8\ (2.31)^2}{7.31 \times 4.20}\sqrt{\dfrac{4.2 \times 9}{6 \times 7.1}}$
6. $\dfrac{384\sqrt{792}\ (0.945)}{\sqrt{7.2} + 8.3\sqrt{5}}$
7. $\dfrac{21.7\ (7.72)^2\ (6.7)^2}{\sqrt{4.67} \times \sqrt{81}}$
8. $\dfrac{2.39\sqrt{6.3}}{(5.1)^2\sqrt{4.7}}$
9. $\dfrac{\sqrt{89.3}\ (7.81)^2}{\sqrt{75} + 8\sqrt{121}}$
10. $\dfrac{75\ (3.81)^2\sqrt{972}}{\sqrt{0.0079}}$

Answers to the Above Exercises.

1. 8.82
2. 33.4
3. 53.1
4. 16.10
5. 6.18
6. 481
7. 2980
8. 0.1063
9. 5.96
10. 382,000

CHAPTER V

CUBES AND CUBE ROOTS

Using "K" Scale

21. Cubes.

Just as 4^2 means 4×4, so 4^3 (read four-cubed) means $4 \times 4 \times 4$. The small number, 3, to the upper right indicates how many 4's (or whatever the number is) must be multiplied together. This small number is called the exponent or power of the number. To illustrate:

$$10^3 = 10 \times 10 \times 10$$
$$(4.7)^3 = 4.7 \times 4.7 \times 4.7$$

It is always possible to multiply these numbers out on the "C" and "D" scales—and in combined operations for complicated calculations, it is sometimes more convenient. However, the "K" scale on the slide rule is designed to give you the cubes of all numbers directly.

The "K" scale is what is called a three-unit logarithmic scale; that is, three complete logarithmic scales of a length which, when placed end to end, equal the length of the single logarithmic scale "D" with which it is usually used. You will note that this "K" scale is so arranged that when the indicator is set to a number on the "D" scale, the cube of that number is given under the hairline on the "K" scale.

ILLUSTRATION: What is the cube of 34.5?

Set indicator to 34.5 on "D"
Under hairline on "K" read **41,100**

To carry out this calculation on the full length scales, do the following:

To 34.5 on "D" set 34.5 on "CI"
Move indicator to 34.5 on "C"
Read **41,100** *under the hairline on "D"*

The reciprocal and folded scales are invaluable in shortening various calculations and one who expects to become proficient in the operation of the slide rule should use these scales as often as possible; as, for instance, dividing a product by the reciprocal of a number as illustrated in the above example gives the same result as multiplying by the number. A tool is of value only when it is used.

22. Cube Roots.

The cube root of a number is a number which when multiplied by itself three times gives the original number. Thus, the cube root of 27 is 3, because $3 \times 3 \times 3$ is 27. The symbol of cube root is $\sqrt[3]{}$ and the cube root of 8000 is written as $\sqrt[3]{8000}$.

The "K" scale is a triple scale, consisting of three identical sections, one following the other. In finding the cube roots of numbers, the "K" scale is considered as a single scale.

The first division of the "K" scale will be referred to as K-LEFT; the second division as K-MIDDLE; and the third division as K-RIGHT. To obtain cube roots of numbers, set the hairline on the number on the "K" scale (see unit below) and read the cube root at the hairline on "D" scale, using:

K-LEFT	for numbers between	1 and 10
K-MIDDLE	" " "	10 and 100
K-RIGHT	" " "	100 and 1000

For numbers greater than 1,000 or less than 1 (unity), proceed as follows:

FIRST: Move the decimal point to the left or right three places at a time until a number between 1 and 1000 is obtained.

SECOND: Take the cube root of this number using K-LEFT, K-MIDDLE, or K-RIGHT as explained above. Place the decimal point after the first figure of this reading.

THIRD: Now move the decimal point in the opposite direction one-third as many places as it was moved in (First) above.

ILLUSTRATION: What is the cube root of 34560?

Move the decimal point to the left three places (one group of three), thus obtaining 34.560. Since the part to the left of the decimal point is between numbers 10 and 100, use the K-MIDDLE scale.

Set indicator to 346 on K-MIDDLE and
Read 3.26 under hairline on "D" scale.

Set decimal point one place $\left[\frac{1}{3}(3) = 1\right]$ *to the right to obtain the answer,* **32.6.**

ILLUSTRATION: What is the cube root of 4,567,000?

Move the decimal point to the left six places (two groups of three), thus obtaining 4.567. Since the part to the left of the decimal point is between 1 and 10, use the K-LEFT scale.

Set indicator to 4567 on K-LEFT and
Read 1.658 under the hairline on "D" scale.

Set decimal point two places $\left[\frac{1}{3}(6) = 2\right]$ *to the right to obtain the answer,* **165.8.**

ILLUSTRATION: What is the cube root of 0.0000315?

Move the decimal point to the right six places (two groups of three), thus obtaining 31.5. Since the part to the left of the decimal point is between numbers 10 and 100, use the K-MIDDLE scale.

Set indicator to 31.5 on K-MIDDLE and
Read 3.16 under the hairline on "D" scale.

Set decimal point two places $\left[\frac{1}{3}(6) = 2\right]$ *to the left to obtain the answer,* **0.0316.**

After a little practice, the steps in determining the location of the decimal point, as well as the correct section of the "K" scale to be used, can be easily determined mentally.

ILLUSTRATION: What is the cube root of 0.00315?

Set indicator to 3.15 on K-LEFT and
Read **0.1466** *under the hairline on "D" scale.*

23. Combined Operations.

The "K" scale can be used to advantage with the other scales to obtain results for various combined operations.

Example 1. Evaluate $23.3 \times \sqrt[3]{87.9}$

To 87.9 on K-MIDDLE set 23.3 on "CI"
Opposite "1" on "C" read **103.5** *on "D"*

Example 2. Evaluate $\dfrac{2.45 \times \sqrt[3]{7.8}}{5.67}$

To 7.8 on K-LEFT set 2.45 on "CIF"
Move indicator to 5.67 on "CIF"
Under hairline on "D" read **0.856.**

Reviewing this last example, the cube root of 7.8 is first multiplied by 2.45 using the "CIF" scale (dividing a product by the reciprocal of a number gives the same result as multiplying by the number), and then this result is divided by 5.67 using the "CIF" scale.

Example 3. Evaluate $\dfrac{34.5 \times 7.93 \sqrt[3]{895}}{(2.38)^3}$

To 895 on K-RIGHT set 2.38 on "CF"
Move indicator to 2.38 on "CIF"
Bring 7.93 on "CI" to hairline
Move indicator to "1" on "C"
Bring 2.38 on "C" to hairline
Move indicator to 34.5 on "C"
Read answer as **195.5** *under hairline on "D"*

Example 4. Evaluate $\dfrac{\sqrt{0.78} \times 8.97 \times \sqrt[3]{54.8}}{4.58 \times 82.1}$

To 54.8 on K-MIDDLE set 4.58 on "C"
Move indicator to 0.78 on B-RIGHT
Bring 8.97 on "CIF" to hairline
Move indicator to 82.1 on "CIF"
Read answer as **0.0799** *under hairline on "D"*

Exercises

1. $\pi\ (63.2)^3$
2. $\sqrt[3]{63.2}\ (\pi)$
3. $7.81\ (2.31)^3$
4. $\sqrt[3]{0.0785}$
5. $\sqrt[3]{92756}$
6. $(0.00312)^3$
7. $\dfrac{(81.2)\ \sqrt[3]{8.1}}{7.2}$
8. $\dfrac{2.45 \times \sqrt[3]{72.8}}{\sqrt{6.3}}$
9. $\dfrac{7.81 + \sqrt[3]{9.71}}{34.2\ \sqrt[3]{752}}$
10. $\dfrac{9.45 \times \sqrt{96.1}}{\sqrt[3]{\dfrac{831}{5.1}}}$
11. $\dfrac{\sqrt[3]{0.0831} \times \sqrt{81.0}}{\pi\ (3.87)^2}$
12. $\sqrt{(2.78)^2 - \sqrt[3]{5.92}}$
13. $\dfrac{(2.81)^3 - \sqrt{8.1}}{(2.03)^2}$
14. $(0.431)\ (0.003)^2\ \sqrt[3]{87.2}$
15. $\dfrac{(\pi)^2\ (1.815)^2}{\sqrt{\pi + 4.18}}$

ANSWERS

1. 794,000
2. 12.52
3. 96.3
4. 0.428
5. 45.3
6. 30.4×10^{-9}
7. 22.6
8. 4.07
9. 0.0320
10. 16.95
11. 0.0835
12. 2.43
13. 4.72
14. 1.715×10^{-5}
15. 12.03

CHAPTER VI

PLANE TRIGONOMETRY

Use of the "S", "T<45", "T>45", and "ST" Scales

24. Fundamental Ideas and Formulas of Plane Trigonometry.

A review of a few of the fundamental ideas and formulas of plane trigonometry is given here to help in understanding the explanation of the use of the "S", "T<45", "T>45", and "ST" scales on your slide rule.

In the right triangle, Figure 25, the corners or angles are labeled A, B, and C. The triangle is referred to as triangle ABC. The sides are labeled a, b, and c, with a opposite angle A, b opposite angle B, and c opposite angle C. For right triangles the 90° angle is labeled C.

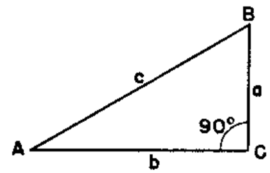


Fig. 25

Referring to this figure, the following definitions and relationships can be written.*

Definitions of the sine, cosine, and tangent:

$$\text{Sine A (written sin A)} = \frac{a}{c} = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\text{Cosine A (written cos A)} = \frac{b}{c} = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

$$\text{Tangent A (written tan A)} = \frac{a}{b} = \frac{\text{opposite side}}{\text{adjacent side}}$$

Reciprocal relations:

$$\text{Cosecant A (written csc A)} = \frac{c}{a} = \frac{1}{\sin A}$$

$$\text{Secant A (written sec A)} = \frac{c}{b} = \frac{1}{\cos A}$$

$$\text{Cotangent A (written cot A)} = \frac{b}{a} = \frac{1}{\tan A}$$

*See any standard text on Plane Trigonometry.

RELATION BETWEEN FUNCTIONS OF ANGLES LESS THAN 90°:

$$\cos A = \sin (90° - A)$$
$$\cot A = \tan (90° - A)$$

Likewise,

$$\sin A = \cos (90° - A)$$
$$\tan A = \cot (90° - A)$$

From a table of functions of angles, the cosine of 35° is given as 0.819152. Looking up the sine of (90° − 35°) or the sine of 55°, we find that it is again 0.819152. You can check these relationships given above in a similar manner.

Complementary angles have their sum equal to 90°. Thus, in the above example, 35° and 55° are complementary angles since their sum is 90°.

RELATION BETWEEN FUNCTIONS OF ANGLES BETWEEN 90° AND 180°:

The definitions of the trigonometric functions given at the beginning of this article apply only to angles between 0° and 90°. More general definitions applying to angles of any size are given in texts on trigonometry. Since we will have to deal with functions of angles between 90° and 180°, a summary of these relationships only will be given here, and one is referred to any text on trigonometry for a complete statement of these definitions.

If A is an angle between 90°, and 180° then the following relationships hold between the functions of these angles:

$$\sin A = \sin (180° - A)$$
$$\cos A = -\cos (180° - A)$$
$$\tan A = -\tan (180° - A)$$

Thus, if the angle A is 123°, we may write:

$$\sin 123° = \sin (180° - 123°) = \sin 57°$$
$$\cos 123° = -\cos (180° - 123°) = -\cos 57°$$
$$\tan 123° = -\tan (180° - 123°) = -\tan 57°$$

From these relationships, the value of the functions of any angle between 90° and 180° can be obtained. These will be used later for the solution of oblique triangles.

RELATION BETWEEN ANGLES OF TRIANGLES

In a right triangle, the sum of the other two angles is 90°. Referring to Figure 25, the sum of A and B equals 90° and the sum of A, B, and C equals 180°.

In equation form:

For a right triangle:

$$A + B = 90° \text{ (where angle C is 90°)}$$

For any triangles:

$$A + B + C = 180°$$

In any triangle as Figure 26, the relation between the sides and the angles can be expressed as shown below:

Law of sines:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Law of cosines:

$$a^2 = b^2 + c^2 - 2bc\ (\cos A)$$
$$\text{Or } b^2 = a^2 + c^2 - 2ac\ (\cos B)$$
$$\text{Or } c^2 = a^2 + b^2 - 2ab\ (\cos C)$$

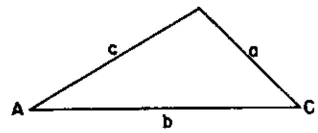


Fig. 26

25. The "S" (Sine) and "ST" (Sine-Tangent) Scales.

The "S" and "ST" scales are two sections of one long scale which, operating with "D", gives the sines of the angles between 0.57° and 90°. The "S" scale represents the scale of sines of angles from 5.74° to 90° (and for cosines of angles from 0° to 84.26°). The "ST" scale represents the scale of sines and tangents of angles from 0.57° to 5.74° and for cosines of angles from 84.26° to 89.43°. Since the value of the sine and tangent of angles below 5.74° is for all practical purposes identical, we can use the same scale for finding either the sine or the tangent for angles below 5.74° and above 0.57°. Thus, the reason for the "ST" scale.

The black numbers on "S" are used for sines and the red numbers for cosines.

The "S" and "ST" scales are so designed and arranged that when the indicator is set to a black number (angle) on the "S" or "ST" scales, the sine of the angle is given under the hairline on the "D" scale.

When using the "S" scale to read the value of sines of any angle, read the left index of "D" as 0.1 and the right index as 1.0. When using the "ST" scale to read the value of the sine of any angle, read the left index of "D" as 0.01 and the right index as 0.1. This is illustrated in Figure 27.

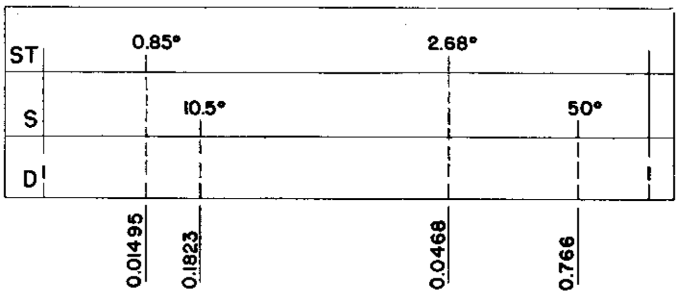


Fig. 27

The "S" scale between the left index (5.74°) and 10° has each degree numbered and the interval between each degree is first divided into ten parts representing 0.1° and each of these ten parts are then divided into two parts—each part representing 0.05°. From 10° to 20°, each degree is numbered and the interval between each degree is divided into ten parts representing 0.1°. Therefore, between the left index (5.74°) and 20° of the "S" scale, you can easily estimate the angles to the nearest 0.01°. From 20° to 30°, the degrees are not numbered except the 25°, but are indicated by a long mark. The interval between the degrees is divided into five parts—each part representing 0.2°. Therefore, between 20° and 30°, you can read to the nearest 0.1°. From 30° to 60°, each ten degrees are numbered and the primary interval between each ten degrees represents 1.0°. Each degree is again divided into two parts by a short mark representing 0.5°. Here you can still estimate to the nearest 0.1°. From 60° to 80°, each ten degrees is marked and numbered and the primary interval between each ten degrees represents 1.0°. With reasonable accuracy, you can estimate to the nearest 0.1°. From 80° to 90°, one can only estimate to the nearest degree.

Example 1. What is the sin 6.75°?
Bring indicator to 6.75 on "S"
Under the hairline read **0.1175** *on "D"*

Example 2. What is the sin 27.584°?
Round this off to either 27.6 or 27.58
The last digit may be estimated fairly well
Set indicator to 27.58 on "S"
Under hairline read **0.463** *on "D"*

Example 3. What is the sin 75.4°?
Set the indicator to 75.4 on "S"
Under the hairline read **0.967** *on "D"*

Example 4. What is the sin 3.45°?
Set indicator to 3.45 on "ST"
Under hairline read **0.0602** *on "D"*

Example 5. What is the sin 0.785°?
Set indicator to 0.785 on "ST"
Under hairline read **0.0137** *on "D"*

EXERCISES

1. Obtain the sine of each of the following angles:

(a) 23.7°	(c) 13.578°	(e) 54.8°	(g) 75.8°	(i) 37.8°
(b) 30°	(d) 23.45°	(f) 87.0°	(h) 45.735°	(j) 20.59°

2. If the cos A = sin (90° — A), determine the cosine of the angles in exercise 1.

3. The sine of various angles are given below. Obtain the angle represented by each.

(a) 0.776	(d) 0.0652	(g) 0.01125
(b) 0.1235	(e) 0.443	(h) 0.678
(c) 0.985	(f) 0.500	(i) 0.563
		(j) 0.0866

Answers to the above exercises.

1.	a. 0.402	c. 0.2348	e. 0.817	g. 0.969	i. 0.613
	b. 0.500	d. 0.398	f. 0.999	h. 0.716	j. 0.352
2.	a. 0.916	c. 0.972	e. 0.576	g. 0.245	i. 0.790
	b. 0.866	d. 0.917	f. 0.0523	h. 0.698	j. 0.936
3.	a. 50.9°	c. 81°	e. 26.3°	g. 0.644°	i. 34.3°
	b. 7.09°	d. 3.74°	f. 30°	h. 42.7°	j. 4.97°

26. The "T < 45" and "T > 45" (Tangent) Scales.

The "T<45" and "T>45" scales are designed to give directly the tangents and cotangents of angles between 5.71° and 84.29°. When the indicator is set to any black number (angle) on the "T<45" and "T>45" scales, the tangent of that angle is given on the "D" scale.

When the indicator is set to any red number (angle) on the "T<45" and "T>45" scales, the cotangent of that angle can be read under the hairline on the "D" scale. Furthermore, because of the reciprocal relationship of $\cot = \frac{1}{\tan}$, when the indicator is set to any black number (angle) on the "T<45" and "T>45" scales, the cotangent of that angle can be read under the hairline on the "DI" scale.

For angles between 0.57° and 5.71°, the tangent and the sine are for all practical purposes almost the same. We can therefore use the "ST" (Sine-Tangent) scale in conjunction with the "D" scale to obtain the tangents and cotangents of angles between 0.57° and 5.71°.

Thus, to determine the tangent of the angle 35.2°, set the indicator to 35.2 on "T<45" and under the hairline on "D", read *0.705* for the tangent. Under the hairline on "DI", read *1.418* as the cotangent.

On slide rules having only one "T" scale, tangents of angles less than 45° are read on the "D" scale while tangents of angles greater than 45° must be read on the "DI" scale. Similarly cotangents of angles less than 45° are read on the "DI" scale while cotangents of angles greater than 45° are read on the "D" scale.

Exercises

1. Determine the tangent of each of the following angles:

(a) 34.5°	(c) 6.905°	(e) 67°	(g) 55.5°	(i) 25.9°
(b) 5.85°	(d) 45°	(f) 7.35°	(h) 37.45°	(j) 80°

2. For each of the angles given in Exercise 1, obtain the cotangent.

3. Determine the angle for which the following numbers are their tangents:

(a) 0.1168	(c) 0.652	(e) 1.567	(g) 0.528	(i) 2.1345
(b) 0.978	(d) 0.500	(f) 4.672	(h) 0.120	(j) 0.5438

4. For each of the numbers given in Exercise 3, obtain the angle for which they are the cotangents.

Answers to the above exercises.

1.	a. 0.687	c. 0.1211	e. 2.356	g. 1.455	i. 0.486
	b. 0.1025	d. 1.000	f. 0.1290	h. 0.766	j. 5.67
2.	a. 1.457	c. 8.26	e. 0.425	g. 0.687	i. 2.059
	b. 9.76	d. 1.000	f. 7.76	h. 1.307	j. 0.1763
3.	a. 6.66°	c. 33.1°	e. 57.44°	g. 27.87°	i. 64.87°
	b. 44.44°	d. 26.6°	f. 77.9°	h. 6.85°	j. 28.56°
4.	a. 83.34°	c. 56.9°	e. 32.56°	g. 62.13°	i. 25.13°
	b. 45.56°	d. 63.4°	f. 12.1°	h. 83.15°	j. 61.44°

27. The Red Numbers on the "S", "T<45" and "T>45" Scales.

The *red numbers* on the "S", "T <45" and "T <45" scales represent the complements of the angles as shown by the corresponding black numbers on these scales. The sum of complementary angles is 90°. Thus, if you set the indicator to the black number 25 (25°) on the "S" or "T <45" scales, you will also be able to read under the hairline the red number 65 (65°). The sum of these numbers is 90.

From this and the fact that $\sin(90° - A) = \cos A$, you can read the functions cosine (and cotangent) directly on the "D" scale by using the *red numbers*. Thus, to obtain the cosine of 65°, do the following:

Set indicator to the red 65 on "S"
Under hairline read **0.423** *on "D"*
Therefore, cos 65° = 0.423

Also, determine the cot. 65°

Set indicator to the red 65 on "T" <45
Under hairline read **0.466** *on "D"*
Cot. 65° = 0.466

The reciprocal function secant (equal to 1/cos A) can be obtained by using the red numbers on "S" and the "DI" scale, since the reciprocal of any number on "D" is given at the hairline on "DI".

ILLUSTRATION: Determine the sec 65°.

Set indicator to the red 65 on "S"
Under hairline read **2.362** *on "DI"*
Sec 65° = 2.362

The reciprocal scale, "DI", can be used to obtain the cotangent of angles by using the black numbers on "T<45" or "T>45". Since the tangent is the reciprocal of the cotangent, it is always possible to convert from one to the other by using the "D" and "DI" scales.

ILLUSTRATION: Again determine the cot 65°.

Set indicator to the black 65 on "T>45"
Read the cotangent as **0.466** *on "DI"*
Read the tangent as **2.145** *on "D"*

28. Summary of Settings on "S", "T<45", "T>45" and "ST" Scales.

As an aid in reviewing the individual settings for the various trigonometric functions, the following summary is given here:

FOR SINES:

0.57° to 5.74° —Read *black* numbers (angles) on "ST" scale to "D" scale (*black* numbers) giving a value between *0.01* and *0.1. Black to Black.*

5.74° to 90° —Read *black* numbers (angles) on "S" scale to "D" scale (*black* numbers) giving a value between *0.1 and 1.0. Black to Black.*

FOR COSINES:

0° to 84.26° —Read *red* numbers (angles) on "S" scale to "D" scale (*black* numbers) giving a value between *0.1* and *1.0. Red to Black.*

84.26° to 89.43°—Use the relationship cos A = sin (90° — A). Read (90° — A) on "ST" scale to "D" scale giving values between *0.01* and *0.1.*

FOR TANGENTS:

0.57° to 5.71° —Read *black* numbers (angles) on "ST" scale to "D" scale (*black* numbers) giving a value between *0.01* and *0.1. Black to Black.*

5.71° to 45° —Read *black* numbers (angles) on "T<45" scale to "D" scale (*black* numbers) giving a value between *0.1* and *1.0. Black to Black.*

45° to 84.29° —Read *black* numbers (angles) on "T>45" scale to "D" scale (*black* numbers) giving a value between *1.0* and *10.0. Black to Black.*

84.29° to 89.43°—Use the relationship tan A = cot (90° — A). Set (90° — A) on "ST" and read answer on "DI" scale (*red* numbers) giving a value between *10.0* and *100.0.*

FOR COTANGENTS:

0.57° to 5.71° —Read *black* numbers (angles) on "ST" scale to "DI" scale (*red* numbers) giving a value between *10.0* and *100.0. Black to Red.*

5.71° to 45° —Read *red* numbers (angles) on "T>45" scale to "D" scale (*black* numbers) giving a value between *1.0* and *10.0. Black to Red.*

45° to 84.29° —Read *red* numbers (angles) on "T<45" scale to "D" scale (*black* numbers) giving a value between *0.1 and 1.0. Black to Red.*

FOR SECANTS:

0° to 84.26° —Read *red* numbers (angles) on "S" scale to "DI" scale (*red* numbers) giving a value between *1.0* and *10.0. Red to Red.*

84.26° to 89.43°—Use the relationship sec A $= \frac{1}{\cos A}$ and cos A = sin (90° — A). Read (90° — A) on "ST" scale to "DI" scale giving a value between *10.0* and *100.0.*

FOR COSECANTS:

0.57° to 5.74° —Use the relationship csc A $= \frac{1}{\sin A}$. Read *black* numbers (angles) on "ST" scale to "DI" scale (*red* numbers) giving a value between *10.0* and *100.0. Black to Red.*

5.74° to 90° —Read *black* numbers (angles) on "S" scale to "DI" scale (*red* numbers) giving a value between *10.0* and *1.0. Black to Red.*

For angles smaller than 0.57° or larger than are shown in the above summary, see article 37 in this chapter covering the functions of small angles.

You will notice that for sine, tangent, and secant (the direct functions), one always reads on like colors, *BLACK* to *BLACK* or *RED* to *RED* numbers (except when you use the relationship of complementary angles). Also, in the same manner for cosine, cotangent, and cosecant (the co-functions), one always reads on opposite colors, *BLACK* to *RED* or *RED* to *BLACK* numbers on the respective scales.

By using the reciprocal relations and the relations between complementary angles as $\cos A = \sin(90° - A)$, any of the six trigonometric functions of an angle can be replaced by a sine or tangent of an angle. Hence, by using these relations, the red scales may be avoided. It is recommended that the student always use the red numbers to avoid subtracting an angle from 90° where possible.

However, if one uses the trigonometric scales infrequently, it is advisable that one employ mainly the sine and tangent.

29. Combined Operations.

Since the "S", "T<45", "T>45", and "ST" scales are placed on the "slide" part of the rule, these scales can be used quite conveniently with the other scales of the rule to solve combined multiplication and division, etc., involving trigonometric functions.

The examples given below illustrate the various types of problems that can be solved using the "S", "T<45", "T>45", and "ST" scales.

Example 1. Evaluate 4.53 sin 12.5°.

This indicates the multiplication of 4.53 times the sine of 12.5°.

Set left end of "S" to 4.53 on "D"
Bring indicator to 12.5 on "S"
Under hairline read **0.982** *on "D"*

Example 2. Evaluate $\dfrac{23.5 \sin 34.7°}{\tan 15.3°}$.

To 23.5 on "D" set 15.3 on "T<45"
Bring indicator to 34.7 on "S"
Under hairline read **48.8** *on "D"*

Example 3. Evaluate $\dfrac{8.34\sqrt{34} \sin 63.0°}{4.23 \tan 42.4°}$.

To 34 on "A-RIGHT" set 4.23 on "C"
Bring indicator to 8.34 on "CF"
Move slide so 42.4 on "T<45" is at hairline
Bring indicator to 63.0 on "S"
Under hairline read **11.20** *on "DF"*

What you have done in the above operations for the solution of Example 3 is this: First, you have divided $\sqrt{34}$ by 4.23 and multiplied this by 8.34 (this result would be at the index on "DF"); second, you have divided by tan *42.4°*; and third, you have multiplied by sin *63.0°*. The answer must, of course, be read on "DF" since the last two operations are done with respect to this scale.

Example 4. Evaluate $\dfrac{67.3 \csc 25° \cos 56°}{\sqrt{5.78} \tan 34.6°}$

Bring 5.78 on "B-LEFT" to left index of "A"
Move indicator to 67.3 on "C"
Bring sin **25°** *(this equals 1/csc 25°) on "S" to hairline*
Move indicator to 56 red *on "S" (this is to* cos **56°** *on "S")*
Bring 34.6 on "T<45" to hairline
Read **53.7** *on "D" opposite right index*

When you have combinations of trigonometric functions involving the reciprocal functions (cosecant, cotangent, and secant), it may help in their solution to write them as 1/sine, 1/tangent, and 1/cosine. For the cosecant of 25° in Example 4, the csc 25° was used on the slide rule as 1/sin 25°.

Example 5. Evaluate $\dfrac{3.42 \times 2.67 \times \sqrt{38.9}}{\sin 80° \times \tan 28° \times 4.08}$

To 3.42 on "D" bring 28 on "T<45"
Move indicator to 38.9 on "B-RIGHT"
Bring 80 on "S" to hairline
Move indicator to 2.67 on "C"
Bring 4.08 on "C" to hairline
Read **26.7** *on "D" at the right index*

The preceding example may be solved in a number of ways using different scales. To illustrate, make the following settings on your rule:

To 38.9 on "A-RIGHT" set 4.08 on "C"
Move indicator to left index of "C"
Bring 2.67 on "CIF" to hairline
Move indicator to 3.42 on "CF"
Bring 28 on "T<45" to hairline
Move indicator to 90 on "S"
Bring 80 on "S" to hairline
Opposite 90 on "S" read **26.7** *on "D"*

This last method is not necessarily shorter. In all of these illustrations, try on your own part to do them in more than one way. This will give you more familiarity with your rule.

Exercises

Evaluate the following problems:

1. $\dfrac{2.45 \cos 36^\circ}{\sin 61.5^\circ}$
2. $\dfrac{3.17 \tan 60^\circ}{\sin 27^\circ}$
3. $\dfrac{45.2\sqrt{7.81}}{\tan 21.5^\circ}$
4. $\dfrac{7.31\ (\pi)\sqrt{45.8}}{31.9 \cot 45^\circ}$
5. $\dfrac{(0.0121) \sin 67^\circ}{8.01 \tan 2.0^\circ}$
6. $\dfrac{13.12 \sin 12.2^\circ}{\csc 38.1^\circ \sqrt{45.3}}$
7. $\dfrac{4.3 \sec 40.8^\circ}{\sqrt{8.31}\ (\tan 5^\circ)}$
8. $\dfrac{\sqrt[3]{95} \sin 45^\circ}{\sqrt{30.3} \tan 19.75^\circ}$
9. $\dfrac{1.015 \cos 31.8^\circ \sin 31.8^\circ}{\sqrt{4.93} \tan 40.9^\circ}$
10. $\dfrac{8.5 \csc 21^\circ \cot 42^\circ}{\sqrt{95.8} \sin 31^\circ \tan 30^\circ}$
11. $\dfrac{(8.5 \times 10^{-5}) \sin 12.75^\circ}{(3 \times 10^{-6}) \sin 16.5^\circ\ (\tan 60^\circ)}$
12. $\dfrac{(0.92)\ (\sqrt{45}) \cot 27^\circ}{5 \tan 18.5^\circ}$

Answers to the above exercises:

1. 2.26	4. 4.87	7. 22.7	10. 9.05
2. 12.09	5. 0.0399	8. 1.634	11. 12.71
3. 321.	6. 0.254	9. 0.236	12. 7.24

30. Solution of Right Triangles.

In many engineering and scientific calculations, it is necessary to determine certain parts of a right triangle having given sufficient information to define the triangle.

CASE I. Given one side "a" and the hypotenuse "c" of a right triangle, determine the side "b" and the angles A and B. (Side "a" is always opposite angle A, side "*b*" is always opposite angle B, and side "c" is always opposite

the angle C, which in this manual is considered as the 90° angle of the right triangle).

Example 1. Find side "b" and angles A and B in a right triangle for which a = 3 and c = 5. See Figure 28.

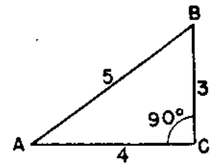


Fig. 28

Solution:

a/c = sin A = cos B.

To 5 on "D" set right index of slide.
Set hairline to 3 on "D".
Under hairline on black "S" scale read A = **36.9°.**
Under hairline on red "S" scale read B = **53.1°.**

b = c sin B

Keep right index of slide still set to 5 on "D".
Opposite 53.1° on black "S" scale read b = **4** *on "D".*

CASE II. Given the hypotenuse "c" and one acute angle B, determine "a", "b", and A.

Example 2. In a right triangle with c = 7.81 and B = 40°, find "a", "b", and A. See Figure 29.

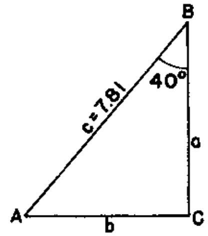


Fig. 29

Solution:

a = c cos B and b = c sin B

To 7.81 on "D" set right index of slide.
Opposite 40° on red "S" scale read a = **5.98** *on "D".*
Opposite 40° on black "S" scale read b = **5.02** *on "D".*
By mental calculation A = 90° − B = **50°.**

CASE III. Given one side "a" and one acute angle A, determine "b" and "c" and the angle B.

Example 3. In a right triangle with a = 17.21 and A = 32.4°, find "b", "c", and B. See Figure 30.

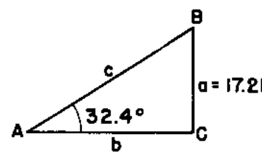


Fig.30

Solution:

By mental calculation
B = 90° − A = *57.6°.*

$$c = \frac{a}{\sin A} \text{ and } b = c \sin B$$

To 17.21 on "D" set 32.4° on black "S" scale.
Opposite right index of slide read c = **32.1.**
Opposite 57.6° on black "S" scale read b = **27.15.**

CASE IV. Given the two sides "a" and "b", determine "c" and the acute angles A and B.

Example 4. Given a = 4 and b = 7, find "c" and A and B.

See Figure 31.

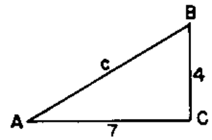


Fig. 31

Solution:

a/b = tan A

Set right index of slide to 7 on "D".
Opposite 4 on "D" read A = **29.8°** *on black "T<45".*
Scale and read B = **60.2°** *on red "T<45" scale.*

$$c = \frac{a}{\sin A}$$

Keep hairline set to 4 on "D" as above.
Bring 29.8° on black "S" scale under the hairline.
Opposite right index of slide read c = **8.05** *on "D".*

31. Solution of Right Triangles by the "Law of Sines."

The law of sines applies to all triangles and is given as

$$\frac{a}{\text{Sin A}} = \frac{b}{\text{Sin B}} = \frac{c}{\text{Sin C}}$$

Or

$$\frac{\text{Sin A}}{a} = \frac{\text{Sin B}}{b} = \frac{\text{Sin C}}{c}$$

The law of sines makes it possible to solve right triangles by proportion on the slide rule. See Chapter 3 on PROPORTION.

Example 1. Given side "a" and angle A as 456 and 34° respectively, determine the hypotenuse and the other leg of the triangle.

See Figure 32.

Write this in the form

$$\frac{456}{\text{Sin } 34°} = \frac{c}{\text{Sin } 90°} = \frac{b}{\text{Sin } (90\text{-A})}$$

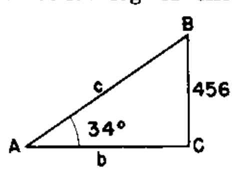


Fig. 32

To 456 on "D" set 34 on "S"
Opposite 90 on "S" read c = **816** *on "D"*
Opposite 56 (90 – 34) on "S" read b = **677** *on "D"*

Example 2. Given a right triangle in which B = 40.8° and c = 78.5 ft., find a, b and A. See Figure 33.

Solution:

A = (90° – B) = 49.2°

To 78.5 on "D" bring 90 on "S"

Opposite 40.8 on "S" read b = **51.3** *on "D"*

Opposite 49.2 on "S" read a = **59.6** *on "D"*

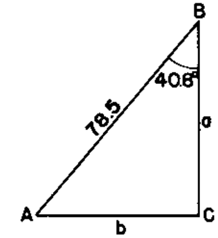


Fig. 33

Reviewing the above two examples—what you have actually done is to divide one number, which is a side of the triangle, by the sine of the angle opposite this side and then multiply this result by the sine of another angle. For instance, the law of sines could be written as follows:

$$c = \frac{a}{\sin A} \times \sin C$$

or any combination of these sides and angles in a similar manner.

31a. Solution of Right Triangles and Vectors by "DI" Scale.

The "DI" scale has useful applications in solving right triangles and vector of problems when any two sides of the triangle are given.

The theory of the "DI" scale as applied to trigonometric and vector problems is as follows:

Figure 34 represents a right triangle for which we can write the following equations:

(1) $\frac{a}{c}$ = sin A, or a = c sin A

(2) $\frac{a}{b}$ = tan A, or a = b tan A

Equate the values of "a" to obtain
(3) a = b tan A = c sin A

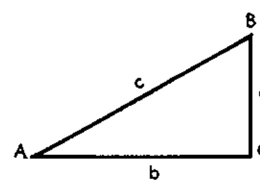


Fig. 34

From Equation (3), the following reciprocal right triangle proportions can be written.

$$(4)\quad \frac{1}{\frac{1}{a}} = \frac{\tan A}{\frac{1}{b}} = \frac{\sin A}{\frac{1}{c}}$$

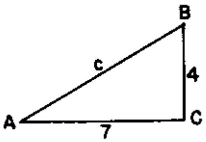


Fig. 34a

EXAMPLE: Solve the right triangle shown in Fig. 34a. In this example the proportion becomes:

$$\frac{1}{\frac{1}{4}} = \frac{\tan A}{\frac{1}{7}} = \frac{\sin A}{\frac{1}{c}}$$

To solve, Set the right index of "S" opposite 4 on "DI"
Move the Hairline to 7 on "DI"
Under the Hairline, on "T <45" (black), read A = *29.8°*
Under the Hairline, on "T <45" (red), read B = *60.2°*
Move the Hairline to 29.8° (black) on "S".
Under the Hairline, on "DI", read c = *8.05*

Note that by using the "DI" scale only one setting of the slide was required.

EXERCISES

In the following exercises for the solution of right triangles C = 90°. Determine the missing parts of the triangle. The law of sines and the proportion principle will be of value in solving these triangles.

1. a = 62.7, A = 30°.
2. b = 31.7, c = 49.8
3. b = 200, A = 68°
4. c = 39.8, a = 12.3
5. c = 423, A = 30°, B = 60°
6. a = 42.8, b = 12.3
7. b = 40.7, c = 59.4
8. a = 12.34, b = 11.97
9. a = 51.2, b = 24.8
10. A = 3.27°, c = 175.8

Answers to the above exercises.

1. B = 60°, b = 108.5, c = 125.4
2. A = 50.4°, B = 39.6°, a = 38.4
3. B = 22°, a = 496, c = 534
4. A = 18°, B = 72°, b = 37.9
5. a = 211.5, b = 367
6. A = 73.97°, B = 16.03°, c = 44.6
7. A = 46.7°, B = 43.3°, a = 43.2
8. A = 45.85°, B = 44.15°, c = 17.18
9. A = 64.1°, B = 25.9°, c = 56.8
10. B = 86.73°, a = 10.05, b = 175.6

32. The Law of Sines Applied to Oblique Triangles.

The same procedure as used for the solution of right triangles by the law of sines can be used for oblique triangles, since the law of sines is applicable to any triangle.

Example 1. Given the oblique triangle in Figure 35 in which c = 43.7 ft., a = 58.9 ft., and A = 35°. Find b, B, and C.

Solution:

$$\frac{c}{\text{Sin } C} = \frac{a}{\text{Sin } A} = \frac{b}{\text{Sin } B}$$

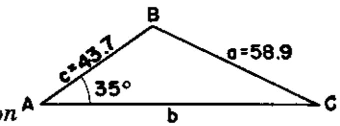


Fig. 35

To 58.9 on "D" set 35 on "S"
Opposite 43.7 on "D" read C = **25.2°** *on "S"*
Since A + B + C = 180°
B = 180° − (35° + 25.2°) = 119.8°
sin 119.8° = sin (180° − 119.8°) = sin 60.2°.
With your slide rule set as above again
Move indicator to 60.2 on "S"
Read b = **89.1** *ft. on "D" under hairline*

Results: B = *119.8°*, C = *25.2°*, and b = 89.1 ft.

Example 2. Given the oblique triangle in Figure 36 in which b = 50.0 ft., a = 4 ft., and B = 68.5°, determine A, C, and c.

Solution:

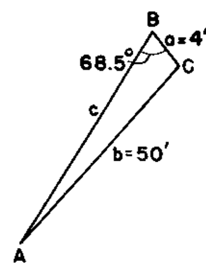


Fig. 36

To 50.0 on "D" set 68.5 on "S"
Move indicator to 4 on "D"
Under hairline read A = **4.27°** *on "ST"*
Move indicator to 72.77 on "S"
Read c = **51.3** *ft. on "D" under the hairline.*
To obtain C = **107.23** *we use the relation C = 180° − (A + B) but sin 107.23° = sin (180° − 107.23°) = sin 72.77°*
72.77° was used above on "S"

Results: A = 4.27°, C = 107.23°, and c = 51.3 ft.

33. Law of Sines Applied to Oblique Triangles (Continued).

When the given parts of a triangle are two sides and an angle opposite one of them, and when the *side opposite* the given angle is *less* than the other given side, there may be *two triangles* which have the given parts. In both the cases solved in the previous article, the side opposite the given angle has been *greater* than the other given side.

Example 1. Given the oblique triangle in Figure 37 in which a = 43.7 ft., c = 58.9 ft., and A = 35°, find b, B, and C. The Figure 37 shows the two possible solutions.

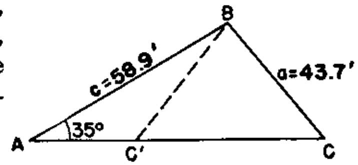


Fig. 37

Solution: (First)

To 43.7 on "D" set 35 on "S"
Opposite 58.9 on "D" read C = **50.5°** *on "S"*
Then B = 180° − (35° + 50.5°) = **94.5°**
With rule set as before
Move indicator to (180° − 94.5°) 85.5 on "S"
Under hairline read b = **76.0** *ft.*

Results of first solution: B = 94.5°, C = 50.5°, and b = 76.0 ft.

Solution: (Second)

The second solution comes in since the sin 50.5° is the same as the sin (180° − 50.5°).
Therefore, in the second solution C = 129.5°.
To 43.7 on "D" set 35 on "S"
Opposite 58.9 on "D" read C = (180° − 50.5°) = **129.5°**.
Now B = 180° − (35° + 129.5°) = **15.5°**
Opposite 15.5 on "S" read b = **20.35** *ft. on "D".*

Results of second solution: B = 15.5°, C = 129.5°, and b = 20.35 ft.

The dotted line in Figure 37 shows the position of the leg "a" for the second solution and for this second solution, the angle C is marked C′.

In this Example 1, it should be noticed that *both solutions* were made with the same setting of the slide rule. This is possible since from trigonometry, we know that *sin A = sin (180° − A)*.

34. Law of Sines Applied to an Oblique Triangle in Which Two Sides and the Included Angle Are Given.

To solve an oblique triangle when two sides and the included angle are given, it is convenient to think of the triangle made up of *two right triangles.* This is illustrated as follows:

Example 1. Given the oblique triangle in Figure 38 in which c = 45.6, b = 67.8, and A = 36.8°, solve the triangle.

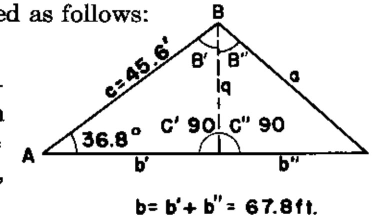


Fig. 38

Solution: The dotted line is drawn from B perpendicular to the base. This forms two right triangles. Call the perpendicular "q".

$$\frac{q}{\sin A} = \frac{45.6}{\sin 90^\circ} = \frac{67.8}{\sin B'} \quad \text{(where } B' = 53.2^\circ\text{)}$$

To 45.6 on "D" set 90 on "S"
Move indicator to 36.8 on "S"
Under hairline read q = **27.3** *on "D"*
Move indicator to 53.2 on "S"
Under hairline read b′ = **36.5** *on "D"*

From the right triangle B″, C, and C″

b″ = 67.8 − 36.5 = *31.3* and

$$\tan C = \frac{q}{b''} = \frac{27.3}{31.3}$$

Set to 31.3 on "D" 27.3 on "C"
Opposite right index of "D" read C = **41.1°** *on "T<45"*
Set to q = 27.3 on "D" 41.1 on "S"
Opposite b″ = 31.3 on "D" read B″ = **48.9** *on "S"*
Opposite 90 on "S" read c″ = **41.5** *on "D"*
B = B′ + B″ = 53.2° + 48.9° = **102.1°**

Results: c″ = a = *41.5*, B = *102.1°*, and C = *41.1°*

If the given angle is greater than 90°, the perpendicular will fall outside the given triangle, but the solution is essentially the same. See Figure 39.

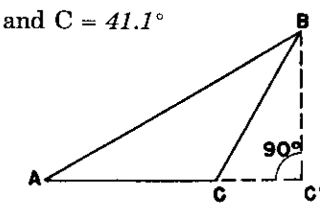


Fig. 39

Exercises

Solve the following oblique triangles. The "ST" scale must be used in Exercises 6, 7 and 10.

1. c = 75, B = 39°, C = 105°
2. a = 12, b = 38, A = 7.8° (Hint): Two solutions
3. b = 7.81, c = 19.75, C = 97°
4. a = 0.7758, b = 0.721, A = 65°
5. b = 90.7, c = 82.1, B = 49.7°
6. a = 4.27, A = 3.75°, C = 100°
7. a = 8, b = 120, B = 60°
8. a = 120, b = 91, A = 58°
9. a = 12.02, b = 7.21, B = 32.7° (Hint): Two solutions
10. b = 3.21, B = 2.39°, C = 103.7°

Answers to the above problems.

1. a = 45.6, b = 48.9, A = 36°
2. *First Solution*: c = 48.6, B = 25.5°, C = 146.7° — *Second Solution*: c = 26.9, B = 154.5°, C = 17.7°
3. a = 17.24, A = 59.9°, B = 23.1°
4. c = 0.722, B = 57.4°, C = 57.6°
5. a = 118.6, A = 86.6°, C = 43.7°
6. b = 63.4, c = 64.3, B = 76.25°
7. c = 123.8, A = 3.31°, C = 116.69°
8. c = 140.2, B = 40.0°, C = 82.0°
9. *First Solution*: c = 13.24, A = 64.2°, C = 83.1° — *Second Solution*: c = 6.97, A = 115.8°, C = 31.5°
10. a = 73.8, c = 74.8, A = 73.91°

35. Law of Cosines Applied to Oblique Triangles in Which Three Sides Are Given.

When the three sides of a triangle are given, we may find the value of one angle by the use of the law of cosines first, and then having one angle known, solve the other angles by means of the law of sines which is easier to use.

Example 1. Given a triangle in which the sides are a = 34.5, b = 52.3, and c = 46.3. Solve the triangle.

Solution:

The law of cosines is

$$a^2 = b^2 + c^2 - 2bc(\cos A)$$

From this we get

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc}$$

$$\text{Cos } A = \sin (90° - A) = \frac{(52.3)^2 + (46.3)^2 - (34.5)^2}{2 \times 52.3 \times 46.3}$$

$$\sin (90° - A) = \frac{3690}{4850}$$

Set to 4850 on "D" 3690 on "C"
Opposite right index of "D" read **49.6** *on "S"*
This is (90° − A)
Therefore, A = **40.4°**

To 34.5 on "D" set 40.4 on "S"
Opposite c = 46.3 on "D" read C = **60.6** *on "S"*
Opposite b = 52.3 on "D" read B = **79** *on "S"*

Results: A = *40.4°*, B = *79.0°*, and C = *60.6°*.
Check: A + B + C = 180°. Thus, 40.4° + 79.0° + 60.6° = 180°.

Exercises

Solve the triangles in the following exercises:

1. a = 20, b = 36.3, c = 39.9
2. a = 3.84, b = 9.06, c = 8.54
3. a = 0.499, b = 0.751, c = 0.704
4. a = 9.75, b = 6.49, c = 5.79
5. a = 2.97, b = 61.0, c = 61.4
6. a = 38.2, b = 45.8, c = 72.8
7. a = 2.19, b = 3.69, c = 3.85
8. a = 87.5, b = 46.4, c = 62.6
9. a = 469, b = 925, c = 633
10. a = 151.0, b = 158.0, c = 123.8

Answers to the above exercises.

1. A = 30°, B = 65°, C = 85°
2. A = 25°, B = 85°, C = 70°
3. A = 40°, B = 75°, C = 65°
4. A = 105°, B = 40°, C = 35°
5. A = 2.75°, B = 80.9°, C = 96.3°
6. A = 27°, B = 33°, C = 120°
7. A = 33.7°, B = 69.2°, C = 77.1°
8. A = 105.7°, B = 30.7°, C = 43.6°
9. A = 27.8°, B = 113.2°, C = 39.0°
10. A = 63.5°, B = 69.3°, C = 47.2°

36. Conversions Between Degrees and Radians.

If an angle is made a central angle of a circle, the number of radians in the angle equals the ratio of the length of the intercepted arc to the length of the radius of the circle. Hence, since an angle of 180° intercepts an arc equal to a semi-circle,

$$180° = \frac{\pi r}{r} = \pi \text{ radians.}$$

Therefore, the following relation can be set up:

$$\frac{\pi}{180} = \frac{\text{R (number of radians)}}{\text{D (number of degrees)}}.$$

Based on this proportion we have the following GENERAL RULE for conversions between degrees and radians:

Set 180 on "CF" to π right on "DF".

Opposite a given number of degrees on "CF" (or "C") read the equivalent number of radians on "DF" (or "D".)

Opposite a given number of radians on "DF" (or "D") read the equivalent number of degrees on "CF" (or "C").

The decimal point is located from a mental estimate.

$$(1 \text{ radian} = \frac{180°}{\pi} = 57.3°).$$

Example 1. How many radians are equivalent to 125.5°?

Set 180 on "CF" to π right on "DF".
Opposite 125.5 on "CF" (or "C") read **2.19** *radians on "DF" (or "D").*

Example 2. How many degrees are equivalent to 5.46 radians?

Set 180 on "CF" to π right on "DF".
Opposite 5.46 on "D" read **313°** *on "C".*

If, in conversions between radians and degrees, an accuracy of ⅙ of 1% is sufficient, then the conversion can be made more simply. For small angles sin A = A (in radians) to a close approximation. Therefore, opposite an angle marked on the "ST" scale, we can read its radian measure A = sin A on the "D" scale. For A = 1° the error in the approximation is 1 part in 200,000. For A = 5.74°, the maximum angle on the "ST" scale, the error is 1 part in 600, or ⅙ of 1% To this accuracy, then, angles marked in degrees on the "ST" scale have their radian values indicated on the "D" scale. *Decimal multiples of angles on the "ST" scale will have radian values which are decimal multiples of the values read on the "D" scale.* Hence the following

GENERAL RULE:

To convert between degrees and radians, to an accuracy of ⅙ of 1% or better, read radians on "D" opposite degrees on "ST", or vice versa.

Example 3. How many radians are equivalent to 125.5°?

Set hairline to 125.5° (or 1.255°) on "ST".
Under hairline on "D" read **2.19** *radians.*

Example 4. How many degrees are equivalent to 5.46 radians?

Close rule and set hairline to 5.46 on "D".
Under hairline on "ST" read **313°**.

Exercises

1. Express the following angles in radians: 3.45°, 76.5°, 45.6°, 0.8°, 48.2°, 346°, 320°, 201°, 308°, and 57.3°.

2. Express the following angles in degrees: 0.089, 2.345, 6.28, 6.34, 5.24, 0.896, 1.0894, 2.34, and 4.72. All given values are in radians.

Answers to the above exercises.

1. 0.0603, 1.336, 0.797, 0.01396, 0.842, 6.04, 5.58, 3.51, 5.38, and 1.00.

2. 5.1°, 134.3°, 360°, 363°, 300°, 51.3°, 62.3°, 134°, and 270°.

37. Sines and Tangents of Small Angles.

The sines and tangents of angles smaller than those given on the "ST" scale can be found by the following approximation:

$$\sin A = \tan A = A \text{ (in radians).}$$

The error in the above approximations is less than 1 part in 10,000 for angles less than 1°.

Therefore, *to find the sine or tangent of an angle less than 1°, find the value of the angle in radians.* Methods for converting an angle from degrees to radians have been given in Section 36. To locate decimal points recall that $1° = \frac{\pi}{180} = 0.01745$ radians.

Example 1. Find sin 0.2°.

Set hairline to 0.2° (or 2°) on "ST".
Under hairline on "D" read sin 0.2° = **0.00349** *on "D".*

If the small angles whose sines or tangents are to be found are given in terms of minutes or seconds, their values in radians may be found by means of the "minute" and "second" marks on the "ST" scale.

$$1' = \frac{1^\circ}{60} = 0.01667^\circ = 0.0003 \text{ radians (approximately).}$$

$$1'' = \frac{1^\circ}{3600} = 0.0002778^\circ = 0.000005 \text{ radians (approximately).}$$

The "minute" mark is placed on the "ST" scale at 1.667° (or 0.01667°) and the "second" mark on the same scale at 2.778° (or 0.0002778°). Below the marks one can read on the "D" scale the values of 1′ and 1″ in radians— 0.000291 and 0.00000485 respectively. To obtain the radian value for any given angle expressed in minutes or seconds one needs merely to multiply the given number of minutes or seconds times the number of radians in one minute or one second by using the gauge marks. The decimal point is placed by making an approximate mental calculation.

Example 2. Express 16′ in radians.

Set left index of slide opposite 16 on "D".
Opposite "minute" mark on "ST" read
16′ = **0.00466** *radians on "D".*

Example 3. Find tan 23″.

Set right index of slide opposite 23 on "D".
Opposite "second" mark on "ST" read tan
23″ = **0.0001114** *on "D".*

Trigonometric functions for angles very near 90° can also be determined by finding the co-named function of the small complementary angles.

Example 4. Find tan 89.75°.

$$\textit{Tan } 89.75^\circ = \cot 0.25^\circ = \frac{1}{\tan 0.25^\circ}.$$

Set hairline to 0.25° (or 2.5°) on "ST".
Under hairline on "DI" read tan 89.75° = **229.**

Exercises

Find the values of the following:

1. sin 3′	3. cot 0.05°	5. sec 18′	7. sin 8.6′	9. cot 0.2′
2. csc 27″	4. tan 36″	6. sin 9.8″	8. tan 0.8′	10. $\left[\frac{\tan 0.34^\circ}{0.0001237}\right]$

Answers to the above exercises.

1. 0.000873	3. 1146	5. 1.000	7. 0.0025	9. 17180
2. 7640	4. 0.0001747	6. 0.0000475	8. 0.000233	10. 48.0

38. Trigonometric Applications.

Applications Involving Vectors: (Also please see Article 31a. for alternate method) In engineering and scientific calculations, there are an infinite number of problems whose solution involves the application of vectors.

A vector is a segment of a straight line with an arrowhead on one end. A vector specifies the magnitude and direction of some quantity. In Figure 40 a vector "R" is shown, and a set of X- and Y-axes has been added. The projection of the vector on the X-axis is called the X-component of the vector. It is denoted by R_x and may be calculated from the formula

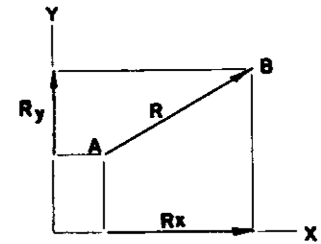


Fig. 40

$$R_x = R \cos A,$$

where R is the magnitude of the vector and A is the angle which the line of the vector would make with the X-axis. Similarly, R_y is the Y-component of the vector, and it may be calculated from the formula

$$R_y = R \sin A.$$

Complex numbers X + jY have two components along perpendicular axes just as do vectors. The magnitude X is the horizontal component of the complex number, and the magnitude Y is the vertical component. The *j* indicates that the magnitude Y is to be laid off along the vertical axis. (Mathematically, $j = \sqrt{-1}$ is the unit imaginary number, and the x- and y-axes are called the real and imaginary axes respectively.)

Example 1. Find the magnitude R and the angle A of the complex number X + jY = 4.3 + j5.7. See Figure 41.

Solution:

$$\frac{X}{Y} = \cot A \text{ and } R = \frac{X}{\sin(90^\circ - A)}$$

Set right index of slide to 5.7 on "D".
Bring hairline to 4.3 on "D".
Under hairline read A = **53.0°** *on red "T<45" scale (A>45° because Y>X.)*

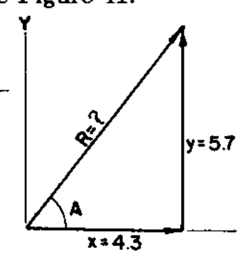


Fig. 41

Note the acute angle 90° − A = 37.0° on black "T<45" scale under hairline.
Bring 37.0° on black "S" scale under the hairline.
Opposite right index of slide read R = **7.15** *on "D".*
Hence, $X + jY = R\underline{/A} = 7.15\underline{/53.0°}$.

The process of finding R and A when X and Y are given is called converting from component to polar form for the complex number, or vector. The process occurs so frequently in problems involving vectors that the following general method will be helpful.

GENERAL METHOD. *To find the magnitude and angle of a vector whose components are known:*

1. Set index of slide to the larger component (X or Y) on "D". Use whichever index will bring the smaller component (Y or X) on "D" opposite some point on the slide scales.
2. Set the hairline to the smaller component (Y or X) on "D".
3. Under the hairline on the black "T<45" (or "ST") scale read the value of the acute angle of the triangle in Figure 41. Write it down.
4. Bring the acute angle on the black "S" scale under the hairline.
5. Opposite the index of the rule read the magnitude of the vector R on "D".
6. Take the angle A of the vector as the acute angle found above or as its complementary angle according to whether $Y < X$ or $Y > X$.

Example 2. An electric circuit has resistance R = 4.3 ohms and reactance X = 3.1 ohms in series. Find the magnitude Z and angle A of the impedance: $Z\underline{/A} = R + jX$. See Figure 41a.

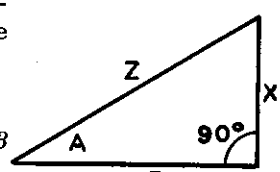


Fig. 41a

Solution:

Set right index of slide to 4.3 on "D".
Bring hairline to 3.1 on "D".
Under hairline read A = **35.8°** *on "T<45".*
Bring 35.8° on black "S" scale under hairline.
Opposite right index of slide read Z = **5.30** *on "D".*
Hence. $R + jX = Z\underline{/A} = \mathbf{5.30}\underline{/\mathbf{35.8}°}$ *ohms.*

APPLICATIONS TO RECTILINEAR FIGURES: The solution of many practical problems is made by working with rectilinear figures. A few typical problems are given below as examples of what can be solved by means of the slide rule.

Example 1. Determine the length of the side CD in the Figure 42.

Solution:

Write $\dfrac{24.7}{\sin 80°} = \dfrac{BD}{\sin 60°}$ and $\dfrac{BD}{\sin 35°} = \dfrac{CD}{\sin 45°}$

To 24.7 on "D" set 80 on "S"
Opposite 60 on "S" read BD = **21.7**
To 21.7 on "D" set 35 on "S"
Opposite 45 on "S" read CD = **26.8**

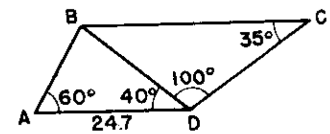


Fig. 42

Or *To 24.7 on "D" set 80 on "S"*
Bring indicator to 60 on "S"
To hairline bring 35 on "S"
Opposite 45 on "S" read CD = **26.8**

In the second method for the solution of Example 1, the intermediate value of BD was not read. This is the only difference in the two methods.

Example 2. A surveyor wants to determine the distance between two inaccessible points A and B and the direction of the line between them. He runs the line CD and finds it 375 ft. in length and bears South 15° East. The angles he measures are as indicated in the Figure 43a.

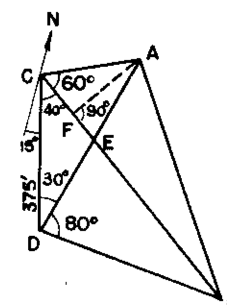


Fig. 43a

Solution: Using the relation between angles in a triangle, determine the various angles of the figure. Next, determine DE and then EB. Next, solve for CE and then EA. Angle AEB is equal to angle CED and thus you can determine two sides (AE and EB) and the included angle.

$\angle$ CED = 180° – (40° + 30°) = 110°
Sin 110° is the same as sin (180° – 110°) = sin 70°
$\angle$ AEB = $\angle$ CED = 110°
$\angle$ DEB = 180° – 110° = 70° which equals $\angle$ CEA
$\angle$ EBD = 180° – (70° + 80°) = 30°
$\angle$ CAE = 180° – (70° + 60°) = 50°

Write:

$$\frac{375}{\sin 110^\circ} = \frac{DE}{\sin 40^\circ} \text{ and } \frac{DE}{\sin 30^\circ} = \frac{EB}{\sin 80^\circ}$$

To 375 on "D" set 70 (180° – 110°) on "S"
Move indicator to 40 on "S"
To hairline bring 30 on "S"
Opposite 80 on "S" read EB = **506** *on "D"*

Likewise:

To 375 on "D" set 70 on "S"
Move indicator to 30 on "S"
To hairline bring 50 on "S"
Opposite 60 on "S" read AE = **225.5** *on "D"*

Drop a perpendicular to CB from A giving AF.

$\angle$ AEF = 70° and sin AEF = $\frac{AF}{225.5}$

From this AF = *212* ft.

$\cos 70^\circ = \frac{EF}{225.5}$ from which

EF = *77.2* ft.
BF = 77.2 + 506 = 583.2

Now $\tan FBA = \frac{FA}{FB} = \frac{212}{583.2}$

$\angle$ FBA = *19.95°*

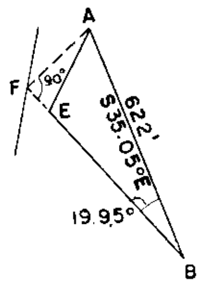


Fig. 43b

To 212 on "D" set 19.95° on "S"
Opposite 90 on "S" read AB = **622** *on "D"*

To determine the direction, add 15° + 40° and subtract 19.95°. This gives 35.05° off of North. Therefore, AB is South 35.05° East.

Results: AB = *622* ft., and AB is *S35.05°E*. See Figure 43b.

Example 3. The diameter of a circle is the base of a triangle having a 7.23 ft. leg. If the diameter of the circle is 14.34 ft., determine the angles of the triangle and the other side.

Solution: The triangle and the inscribing circle are shown in Figure 44.

The side AB is the diameter and angle C is 90°.

Use the law of sines to solve this triangle.

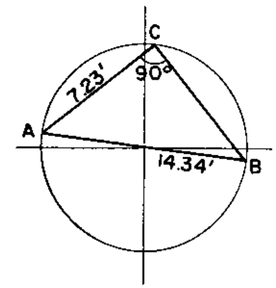


Fig. 44

To 14.34 on "D" set left index of "C"
This is the same as placing 90 on "S" opposite 14.34 on "D"
Opposite 7.23 read B = **30.3°**
Opposite (90° – 30.3°) = 59.7° on "S" read
a = **12.37** *on "D"*

For the last step 59.7 is off the rule with the setting given. To obtain the reading, you must bring the right index of "C" (90 on "S") to 14.34 on "D". Now opposite *59.7°* on "S", you can read a = **12.37** on "D".

Exercises

It is recommended that right angle vector problems also be solved using the alternate method explained in Article 31a.

1. Determine the unknown angles and the unknown magnitudes of the vectors of (a) Fig. 45, (b) Fig. 46, (c) Fig. 47, and (d) Fig. 48.

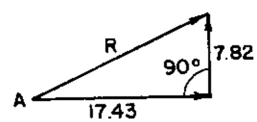


Fig. 45

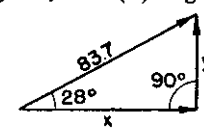


Fig. 46

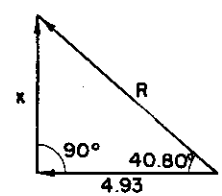


Fig. 47

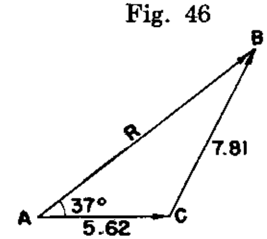


Fig. 48

2. The rectangular components of a vector are + 13.45 feet horizontally and + 7.45 feet vertically. Determine the magnitude of the vector and the angle it makes with the horizontal.

3. Find the horizontal and vertical components of a vector having a magnitude of 56.7 pound, and making an angle of 19.5° with the horizontal.

4. A 34.5 pound vector and an unknown vector "r" have as a resultant a 67.5 pound vector which makes a 32° angle with the 34.5 pound vector. Determine the unknown vector "r". See Figure 49.

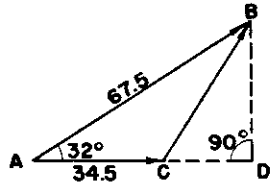


Fig. 49

5. Determine the magnitude and the angle of the vector (measured counter-clockwise from the positive X-axis) representing the complex numbers—3.57 + J 5.67.

6. Determine the length of the unknown side (marked with a letter) in the rectilinear figures shown in (a) Figure 50, (b) Figure 51, and (c) Figure 52.

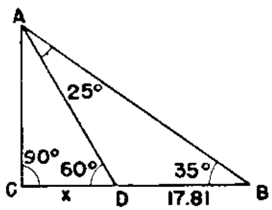


Fig. 50

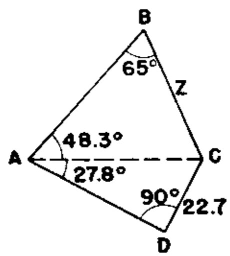


Fig. 51

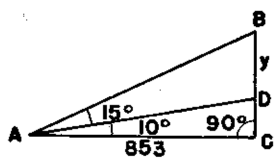


Fig. 52

7. AB is vertical in Figure 53 and represents a tower on a hill. The line CD was measured and found to be 1248′ in length. The angles were measured and are as given in the figure. Determine the height of the tower AB.

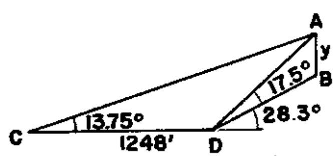


Fig. 53

Answers to the above exercises:

1. (a) A = 24.15°, R = 19.14 (b) Y = 39.3, X = 73.8 (c) X = 4.27, R = 6.52 (d) B = 25.65, C = 117.35°, R = 11.54

2. R = 15.39, A = 29° 3. Y = 18.86 lb., X = 53.4 lb. 4. R = 42.3 lb. 5. A = 122.2°, R = 6.70

6. (a) X = 12.08 (b) Z = 40.1 (c) Y = 248

7. Y = 191 ft.

CHAPTER VII

EXPONENTS, LOGARITHMS, AND THE "L" SCALE

39. Exponents.

In Chapter 4, the squares of numbers were obtained by the use of the following scales: "C", "D", "A", and "B". The notation used was

$$4^2 = 4 \times 4 \text{ or } 16$$

This small number to the upper right of the 4 is called the exponent. If 4 is to be cubed, it is written as 4^3; and in this case, the exponent is 3. Another term used for "*exponent*" is the "*power*" of the number.

Four raised to the second power is 4^2, or four raised to any power "a" is 4^a. The "4" in this case is called by definition the "base". Thus, any number can be a so called "base".

A short table using 10 as a base follows:

$$10^1 = 10 = 10$$
$$10^2 = 10 \times 10 = 100$$
$$10^3 = 10 \times 10 \times 10 = 1{,}000$$
$$10^5 = 10 \times 10 \times 10 \times 10 \times 10 = 100{,}000$$

From this we see that $100 \times 1{,}000 = 100{,}000$

Since 100 is 10^2 and 1000 is 10^3, we may write

$$10^2 \times 10^3 = 10^{2+3} = 10^5$$

In this manner, we are using the *addition* of the exponents to obtain our results. Thus, in the *multiplication* of exponential terms TO THE SAME BASE, *add* the exponents for the result.

$$100{,}000 \div 100 = 1{,}000 \text{ or } \frac{10^5}{10^2} = 10^{5-2} = 10^3$$

From this and the above table, it is seen that in order to *divide* exponential terms TO THE SAME BASE, it is only necessary to *subtract* their exponents.

ILLUSTRATION: What is the value $\frac{2^7}{2^4}$?

$$\frac{2^7}{2^4} = \frac{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2}{2 \times 2 \times 2 \times 2} = 2^3$$

$$\text{or } \frac{2^7}{2^4} = 2^{7-4} = 2^3 \text{ (the same as before)}$$

In this illustration, the number "2" is used as the base.

40. Negative Exponents.

If 10^3 is divided by 10^5, the result would be 10^{3-5} or 10^{-2}. This indicates that the result is $\frac{1}{100}$. Thus, using 10 as a base and for any negative exponent, the result can be indicated by $1 \div 10^{+a}$, where "a" was the negative exponent.

ILLUSTRATION: What is $10^2 \div 10^7$?

$10^{2-7} = 10^{-5}$ or this may be written as

$$10^{2-7} = 10^{-5} = \frac{1}{10^5}$$

41. Notation Using the Base "10".

It is often convenient to change a number by either multiplying or dividing it by 10 to some exponent.

ILLUSTRATION: Change the number 30,000,000 to a more convenient form.

Divide this by 10^6 and write the number as 30×10^6

Or divide by 10^7 and write the number as 3×10^7

Change the number 0.000065 to a more convenient form.

Multiply this number by 10^5 and write the number as 6.5×10^{-5}

In the first illustration, the exponent of 10 is positive; and this indicates that the actual number of digits to the right of the number is the same as the exponent of 10. In the second illustration, the actual number of places to the left of the decimal as indicated by 10^{-5} is 5.

In each case, the number of places through which the decimal point moves is equal to the exponent of ten.

ILLUSTRATION: Evaluate $3450 \times 732 \times 0.032$

First, this can be changed to

$3.45 \times 10^3 \times 7.32 \times 10^2 \times 3.2 \times 10^{-2}$

Again, write it as

$3.45 \times 7.32 \times 3.2 \times 10^{3+2-2}$

To 7.32 on "D" set 3.2 on "CI"
Bring the indicator to 3.45 on "C" and
Read the answer as **80.8** *on "D" under* ***hairline.***
Correct answer is then 80.8 × 10^3 or **80,800**

42. Logarithms.

Logarithms are nothing more than exponents. A base is selected and the logarithm of any number to this base is just the *exponent* of the base that will give the original number. Usually the base 10 is selected and most all tables of logarithms are made to this base.

By definition the

Log 25 to the base 10 is 1.398

This being

$10^{1.398} = 25$

To either multiply or divide by logarithms, one adds or subtracts the logarithms of the numbers. From the above, one can see that this is the same as *adding* or *subtracting* the exponents of 10. This makes a convenient method of multiplying or dividing in complicated calculations.

43. The "L" (Logarithmic) Scale.

As stated in the first chapter, the logarithm of a number has two parts—the part to the *right* of the decimal called the "*mantissa*" and the part to the *left* of the decimal called the "*characteristic*".

The characteristic of the logarithm of a number greater than one is obtained by inspection since it is defined as a number equal to one less than the number of digits in the original number. The characteristic for the logarithm of 456.0 is 3-1 or 2. The mantissa can be found on the "L" scale.

The "L" scale is so designed that when the hairline is placed to any number on the "C", scale the mantissa of the logarithm of that number is shown on the "L" scale.

ILLUSTRATION: What is the logarithm of 456.0?

Set the indicator to 456 on "C"
Read 0.659 on the "L" scale—this is the mantissa.
By inspection, the characteristic is 2.
Therefore, the logarithm of 456 is **2.659.**

The characteristic of a number less than 1 is negative and is numerically one greater than the number of zeros immediately following the decimal point. See Article 4 in Chapter One.

ILLUSTRATION: What is the logarithm of 456?

$456 = 4.56 \times 10^2$. Characteristic is 2.
Set hairline to 4.56 on "C".
Under hairline on "L" read 0.659, the mantissa.
Therefore, log 456 = **2.659.**

ILLUSTRATION: What is the logarithm of 0.0752?

$0.0752 = 7.52 \times 10^{-2}$. Characteristic is -2.
Set hairline to 7.52 on "C".
Under hairline on "L" read 0.8761, the mantissa.
Therefore, log 0.0752 = −2 + 0.8761 = **8.8761 − 10.**

44. Calculations by Logarithms.

The logarithm of a number to the base 10 is defined as the exponent of 10 that will give the number. Thus,

$$10^2 = 100$$

Therefore, the logarithm of 100 is 2 because 10 raised to the second power gives 100.

Likewise, Log 34.5 = 1.5378. This means

$$10^{1.5378} = 34.5$$

Since the logarithms as given on the "L" scale are all to the base ten, one can multiply and divide by obtaining the logarithms of the numbers and then either adding or subtracting the logarithms depending upon whether you want to multiply or divide. The addition and subtraction of the logarithms is the same as the addition or subtraction of exponents as explained in article 39—the base being 10 in this case.

ILLUSTRATION: Evaluate $\dfrac{34.5 \times 9716}{3.24}$

Obtain the log 34.5 = 1.5378
Obtain the log 9716 = 3.9875
Their sum is 5.5253
Obtain the log 3.24 = 0.5106
Their difference is 5.0147

Set the indicator to 0.0147 on "L" scale
Under hairline read **1.034**
Characteristic is 5; therefore, the answer is
1.034×10^5 = 103,400.

This indicates a method of calculating problems as above, but as this can be done easier with the "C", "D", etc., scales, the "L" scale is used primarily when numbers with exponents are to be either multiplied or divided.

ILLUSTRATION: Evaluate $\dfrac{(3.24)^{2.5}\,(45.6)}{(34.5)^{1.35}}$

Analyzing this computation: The log $(3.24)^{2.5}$ is equal to $2.5 \times$ log 3.24 and the log $(34.5)^{1.35}$ is $1.35 \times$ log 34.5. Therefore, obtain the log of these numbers and multiply them by their respective exponents.

SOLUTION: Log 3.24 as read on "L" scale is 0.511
Log 34.5 as read on "L" scale is 1.538
Log 45.6 as read on "L" scale is 1.659

Set 0.511 on "CI" to 2.5 on "D"
Read **1.278** *on "D" opposite 1 on "C"*

Set 1 on "C" to 1.35 on "D"
Opposite 1.538 on "C" read **2.075** *on "D"*

$2.5 \times$ log 3.24 = 2.5×0.511 = 1.278
Log 45.6 = 1.659
Their sum is 2.937
Subtract $1.35 \times$ log 34.5 2.075
This difference is 0.862

Set hairline to 0.862 on "L" scale
Under hairline read **7.27** *on "D".*

The "L" scale, can be used in the same manner as a table of logarithms. This was done in the above illustration.

Exercises

1. By use of the "L" scale, determine the logarithms of the following numbers: 3.45, 34.5, 34500, 52.9, 0.00845, 0.95638, 4.56, 34.92, 5.6638, 0.056638, 78.48×10^{-2}.

2. Evaluate the following problems:

(a) $4^{2.13}$
(b) $3.45 \times (8.4)^{0.8}$
(c) $(23.5)^{2.1} \times \sqrt{3.78}$
(d) $(7.32)^{1/2}\,(34.7)^{0.35}$
(e) $\dfrac{34.5 \times \sqrt{8.1}}{(3.75)^{0.9}}$
(f) $10^{2.50}$
(g) $\dfrac{10^{0.75} \times 10^{4.5}}{10^{5.25}}$
(h) $10^{3.2} \times 10^{-4.2} \times 10^{6.25}$
(i) $(2.34 \times 10^{-5})\,(54.7 \times 10^3)$
(j) $3^{0.45} \times 3^{-1.07} \times 3^{0.82}$

Answers to the above exercises

1. 0.538, 1.538, 4.538, 1.724, 7.927-10, 9.981-10, 0.659, 1.543, 0.753, 8.753-10, 9.895-10.

2. (a) 19.2 (b) 18.92 (c) 1480 (d) 9.37 (e) 29.8 (f) 316 (g) $10^0 = 1$ (h) $10^{5.25}$ = 178,000 (i) 1.28 (j) $3^{0.2} = 1.246$

CHAPTER VIII

THE LOG LOG SCALES

45. The "LL" Scales.

The most frequent use of the Log Log (LL) scales is to find the powers and roots of numbers. Engineering and scientific calculations frequently involve non-integral powers and roots of quantities, and they often involve powers of e and logarithms of numbers to the base e, where $e = 2.71828\ldots$ is the base of *natural* logarithms. With the "LL" scales the computation of $(1.23)^{1.84}$ becomes as simple as multiplying 1.23×1.84, and the evaluation of $\log_e 102.5$ becomes as easy as finding $\frac{1}{102.5}$.

Example 1. Find $(1.23)^{1.84}$. (See Figure 54 below).
Set left index of slide opposite 1.23 on "LL2" scale.
Opposite 1.84 on "C" read $(1.23)^{1.84} =$ **1.464** *on "LL2"*

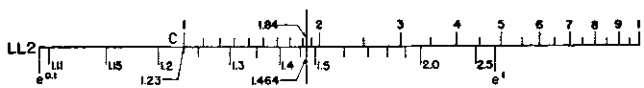


Fig. 54

Example 2. Find $\log_e 102.5$.
Set hairline to 102.5 on "LL3" scale.
Under hairline on "D" scale read $\log_e 102.5 =$ **4.63.**

The "LL" scales are designed to solve problems of the following types:

1. $Y = X^n$. Given X and n, find Y.
2. $n = \log_x Y$. Given X and Y, find n.

The above types of problems can be solved with one setting of the slide, or merely by a setting of the hairline if $X = e = 2.71828\ldots$

GENERAL RULES:

1. *To find* $Y = X^n$, *set index of slide opposite X on the appropriate "LL" scale. Opposite n on the "C" (or "B") scale read Y on the appropriate "LL" scale.*

2. *To find* $n = \log_x Y$, *set index of slide opposite X on the appropriate "LL" scale. Opposite Y on the appropriate "LL" scale read n on the "C" scale.*

Example 3. Find the amount A of \$100 invested for ten years at 5%, compounded semi-annually.

$$A = \$100\left(1 + \frac{0.05}{2}\right)^{20}$$

Set left index of "C" scale to 1.025 on "LL1".
Opposite 20 on "C" read A = **163.86** *on "LL2".*

Example 4. A plate of glass transmits 0.88 of the light incident on it. Find the number of plates n necessary to cut the transmitted light down to 0.50 or less.

$0.50 \geq 0.88^n$

Set left index of "C" scale to 0.88 on "LLO2".
Opposite 0.50 on "LLO2" read 5.42 on "C".
Hence **6** *plates of glass will be used.*

The six "LL" scales may be considered in two groups. First, the "LL1", "LL2" and "LL3" scales which cover numbers greater than 1.00 (from 1.010 to 22026). Second, the "LLO1", "LLO2" and "LLO3" scales which cover numbers less than 1.00 (from 0.00005 to 0.9905). The detailed techniques for using these groups of scales are explained in separate sections below.

46. The "LL1", "LL2", and "LL3" Scales—For Numbers Greater than Unity.

In Figure 55 the "LL1", "LL2", and "LL3" scales are shown as sections of one long scale representing numbers from 1.010 to 22,026. They are aligned with three sections of the "D" scale placed end to end.

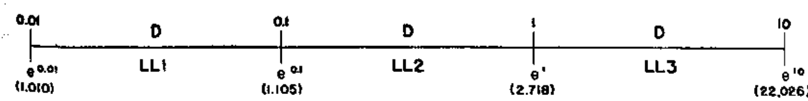


Fig. 55

As shown in Figure 55 the number $e = 2.71828\ldots$ lies at the junction of the "LL2" and "LL3" scales opposite an index of the "D" scale. Because of this alignment of the scales *a number on an "LL" scale is equal to e raised to the power opposite that number on* *the "D" scale.* The range of numbers and of powers of e covered by each of the three scales under consideration is indicated in the table below.

Scale	Range of Numbers	Range of Powers of e
LL1	1.010 to 1.105	0.01 to 0.10
LL2	1.105 to 2.718	0.10 to 1.0
LL3	2.718 to 22,026	1.0 to 10

On the actual slide rule the three sections of the "LL" scale have been placed over one another so that they are aligned with the "D" scale on the body of the rule. Hence, an exponent read on "D" must correspond, in location of the decimal point, to the "LL" scale on which the number is read.

Example 1. Find $e^{0.5}$ and e^{5}.

Set hairline to 5 on "D"
Under hairline on "LL2" read $e^{0.5}$ = **1.649.**
Under hairline on "LL3" read e^{5} = **148.**

A. Finding powers of numbers greater than unity.

In Example 1 note that $e^{0.5}$ is less than e while e^{5} is greater than e. These results illustrate a GENERAL RULE which is very helpful in finding powers of numbers. *When any given number greater than unity is raised to a power, it will yield a result which is greater or less than the given number according to whether the exponent is greater or less than 1.00.*

Let us now develop methods for finding powers of any number greater than unity. The construction of the "LL" scales is such that if an index of the "C" scale is placed opposite a given number on the "LL1", "LL2", or "LL3" scale, then a power of the given number may be found on the appropriate "LL" scale opposite the indicated exponent on the "C" scale. Look back at Figure 54, which illustrates the setting of the rule for evaluation of $(1.23)^{1.84}$ = 1.464. Since the exponent was greater than 1.00, we worked toward the right along the "LL2" scale and found a result (1.464) which was greater than the given number. If the exponent were less than 1.00, the result would be less than 1.23.

Example 2. Evaluate $(1.23)^{0.8}$

Set right index of "C" opposite 1.23 on "LL2".
Opposite 0.8 on "C" read $(1.23)^{0.8}$ = **1.18** *on "LL2".*

In finding a power of a given number:

(a) if the exponent is greater than 1.00, the result will lie to the right of the given number along the chain of "LL" scales in Figure 55, and

(b) if the exponent is less than 1.00, the result will lie to the left of the given number.

For any two numbers separated by one scale length along the chain of "LL" scales, the number to the right is the 10th power of the number to the left, or the lefthand number is the 0.1 power of the righthand number. On the slide rule such numbers lie opposite each other on different "LL" scale.

Sometimes in finding powers of numbers the result lies off the "LL" scale on which the given number is located. In such a case the "LL" scales are treated as one long scale (Figure 55), and the slide is set to read the result on the proper scale. For instance, let us find $(1.5)^5$. (See Figure 55a below.) If we set the left index of "C" to 1.5 on "LL2" in the usual manner, the value of $(1.5)^5$ would be read (opposite 5 on "C") on the fictitious dotted "LL3" scale extending rightward from the "LL2" scale. Since, on the actual rule the dotted "LL3" scale has been slid left one scale length, we can find the value of $(1.5)^5$ by sliding the "C" scale back one scale length in Figure 55a and reading $(1.5)^5$ on "LL3" opposite 5 on "C".

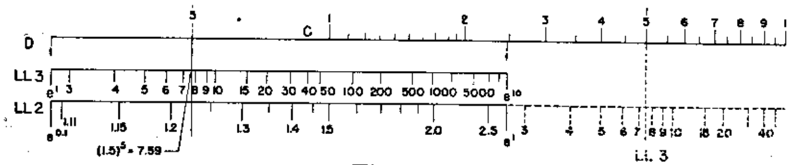


Fig. 55a

Set right index of "C" to 1.5 on "LL2".
Opposite 5 on "C" read $(1.5)^5$ = *7.59 on "LL3".*

Example 3. Evaluate $(7)^{0.12}$ and $(7)^{1.2}$.

Set left index of "C" opposite 7 on "LL3".
Opposite 0.12 on "C" read $(7)^{0.12}$ = **1.263** *on "LL2".*
Opposite 1.2 on "C" read $(7)^{1.2}$ = **10.3** *on "LL3".*

Example 4. Find the amount A of $100 invested for 30 years at 4%, compounded semi-annually.

A = $100 $(1 + 0.02)^{60}$

Set right index of "C" opposite 1.02 on "LL1".
Opposite 60 on "C" read A = $328 on "LL3".

If the exponent is given in fractional form, the settings of the slide rule are similar to those used in multiplying a number by a fraction.

Example 5. Evaluate $e^{\frac{1}{2}}$.

Set 2 on "C" to e on "LL2".
Opposite left index of slide read $e^{\frac{1}{2}}$ = **1.649** *on "LL2".*
(The result checks with $e^{0.5}$ = 1.649 from Example 1.)

Example 6. Evaluate $(27)^{\frac{2}{3}}$.

Set 3 on "C" opposite 27 on "LL3".
Opposite 2 on "C" read $(27)^{\frac{2}{3}}$ = **9** *on "LL3".*

Example 7. Evaluate $\sqrt[7]{(1.2)^2} = (1.2)^{\frac{2}{7}}$.

Set 7 on "CF" opposite 1.2 on "LL2".
Opposite 2 on "CF" read $(1.2)^{\frac{2}{7}}$ = **1.0535** *on "LL1".*

B. Finding logarithms of numbers greater than unity.

The "LL1", "LL2", and "LL3" scales are well adapted to finding logarithms of numbers to any base, but especially to the base *e*. The logarithm of a number to a given base is simply the exponent to which one must raise the base to yield the number. Thus, if $Y = e^n$, then $n = \log_e Y = \ln Y$. Logarithms to the base *e* are called *natural* logarithms and will be written ln Y to distinguish them from common logarithms (to the base 10), which will be written log Y. If other bases are used, they will be indicated. Thus,

(1) if $Y = e^n$, then $n = \ln Y$,
(2) if $Y = 10^n$, then $n = \log Y$, and
(3) if $Y = X^n$, then $n = \log_X Y$.

Because of the alignment of the "LL" and "D" scales *a value read on "D" is the natural logarithm of the opposed number on the corresponding "LL" scale.*

Example 8. Find ln 21.3.

Set hairline to 21.3 on "LL3".
Under hairline on "D" read ln 21.3 = **3.06**.

The "L" scale is used to determine the mantissas of common logarithms. The characteristics may be found by reading off the exponent of 10 after the given number has been expressed as a product of a number between 1 and 10 multiplied by a power of 10.

Example 9. Find log 230.

$230 = 2.3 \times 10^2$. Characteristic is 2.
Set hairline to 2.3 on "D".
Under hairline on "L" read 0.362.
log 230 = **2.362**.

Example 10. Find log 0.00872.

$0.00872 = 8.72 \times 10^{-3}$. Characteristic is -3.
Set hairline to 8.72 on "D".
Under hairline on "L" read 0.940.
log 0.00872 = −3 + 0.940 = **7.940 − 10**.

TO FIND LOGARITHMS TO ANY BASE other than *e* or 10, slide rule is set as in determining the power of a given number.

$Y = X^n$. Given Y and X, find $n = \log_X Y$.
Set index of slide to X on an "LL" scale.
Opposite Y on an "LL" scale read n on "C" and place its decimal point properly.

Example 11. Find $\log_{1.5} 2$.

Set left index of slide opposite 1.5 on "LL2".
Opposite 2 on "LL2" read $\log_{1.5} 2$ = **1.71** *on "C".*

Example 12. How long must a sum of money be invested in order to double itself, if the interest is 3%, compounded annually?

$(1.03)^n = 2$.
Set left index of slide to 1.03 on "LL1".
Opposite 2 on "LL2" read n = **23.5** *years on "C".*

Example 13. Find $\log_{20} 1.2$.

$1.2 = 20^n$.
Set right index of "C" opposite 20 on "LL3".
Opposite 1.2 on "LL2" read n = **0.061** *on "C".*

Similar settings may be used to find an unknown base.

$X^n = Y$. Given Y and n, find X.
Set n on "C" to Y on an "LL" scale.
Opposite index on "C" read X on the proper "LL" scale.

Exercises

Evaluate the following expressions. Use the "LL3", "LL2", and "LL1" scales as may be required. Determine "X" in Exercise 9, 10, 11, and 12.

1. $e^{6.3}$
2. $(6.31)^{2.15}$
3. $e^{0.014}$
4. $(3.16)^{0.75}$
5. $(3.16 \times \pi)^{2.7}$
6. $\left(\frac{9.20 \times 3.8}{18.6}\right)^{0.712}$
7. $(1.319)^{\frac{2.75}{3.21}}$
8. $e^{\frac{1.82}{0.5}}$
9. $10.7^{x} = 92.5$
10. $(X)^{2.81} = 1.218$
11. $(1.123)^{x} = 3.27$
12. $(X)^{2.3} = 85.9$
13. $\sqrt[2.7]{81}$
14. ln 67
15. log 0.171
16. ln 1.014
17. $\log_2 9$
18. log 367
19. $\log_3 243$
20. $\log_{\pi} 1.331$

Answers to the above exercises.

1. 545
2. 52.5
3. 1.0141
4. 2.37
5. 490
6. 1.565
7. 1.268
8. 38.2
9. 1.91
10. 1.0726
11. 10.20
12. 6.92
13. 5.09
14. 4.20
15. 9.233 – 10
16. 0.0139
17. 3.17
18. 2.564
19. 5
20. 0.25

47. The "LLO1", "LLO2" and "LLO3" Scales—For Numbers Less than Unity.

In Figure 56, the "LLO1", "LLO2" and "LLO3" scales are shown as sections of one long scale representing numbers from 0.00005 to 0.9905. They are aligned with three scale lengths of the "D" scale placed end to end.

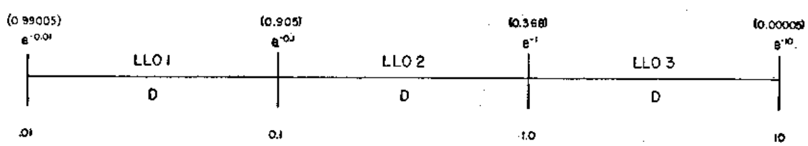


Fig. 56

Because of the arrangement of scales, a number on the "LLO" scales is equal to $e^{-1}\left(\frac{1}{e}\right)$ raised to the power indicated opposite that

number on the "D" scale. The range of numbers and of powers of e^{-1} $\left(\frac{1}{e}\right)$ covered by each of the three scales under consideration is indicated in the table below:

Scale	Range of Numbers	Range of Powers of e^{-1}
LLO1	0.99005 to 0.905	0.01 to 0.1
LLO2	0.905 to 0.368	0.1 to 1
LLO3	0.368 to 0.00005	1.0 to 10.0

On the actual slide rule, the three sections of the "LLO" scales have been placed over each other so that they are aligned with the "D" scale on the body of the rule. Hence, an exponent on the "D" scale must correspond, in location of the decimal point, to the "LLO" scale on which the number is read. For example, if the exponent in the problem is -5, the "LLO" scale for which the value of *e* varies from −1 to −10 must be used, namely the "LLO3" scale.

Example 1. Find $e^{-0.5}$, e^{-5} and $e^{-0.05}$

Set hairline to 5 on "C".
Under hairline on "LLO2" read $e^{-0.5}$ = **0.607.**
Under hairline on "LLO3" read e^{-5} = **0.0067.**
Under hairline on "LLO1" read $e^{-0.05}$ = **0.9512.**

For the convenience of the user, the limiting values of the exponent of *e* for each of the "LLO" scales have been marked on the slide rule.

A. Finding powers of numbers less than unity.

In finding powers of numbers less than unity remember the following GENERAL RULE. *When any given number less than unity is raised to a power, it yields a result which is less than or greater than the given number according to whether the exponent is greater or less than 1.00.* The "LLO1", "LLO2" and "LLO3" scales are laid out with numbers decreasing from left to right. Hence, in finding a power of a given number:

(a) if the exponent is greater than 1.00, the result will lie to the right of the given number along the chain of "LLO" scales in Figure 56, and

(b) if the exponent is less than 1.00, the result will lie to the left of the given number.

For any two numbers separated by one length of the "D" scale (the distance between adjacent indices), the number to the right is the tenth power of the number to the left on the "LLO" scale; or the lefthand number is the one-tenth power of the righthand number. Numbers on the "LLO3" scale are the hundredth power of the numbers opposite them on the "LLO1" scale.

Calculations using the "LLO" scale for numbers less than unity are made by settings quite analogous to those described above for the "LL" scales for numbers greater than unity.

Example 2. Evaluate $(0.8)^3$.

Set left index of "C" opposite 0.8 on "LLO2".
Opposite 3 on "C" read $(0.8)^3$ = **0.512** *on "LLO2".*

Example 3. Evaluate $(0.45)^{0.057}$.

Set right index of "C" opposite 0.45 on "LLO2".
Opposite 0.057 on "C" read $(0.45)^{0.057}$ = **0.9555** *on "LLO1".*

Example 4. What amount P invested at the present time at 3% interest, compounded semi-annually, will amount to $1 in 25 years?

$$P = (1.015)^{-25} = \left(\frac{1}{1.015}\right)^{25}.$$

Set hairline to 1.015 on "C".

Under hairline on "CI" read $\frac{1}{1.015} = 0.985$.

Set left index of "C" to 0.985 on "LLO1".
Opposite 25 on "C" read P = **$0.686** *on "LLO2".*

Example 5. The current in an electric circuit decreases by a factor of $\frac{1}{e}$ every 2.5 seconds. If the current starts at 10 amperes, what will be its value I after 6 seconds?

$$I = 10\left(\frac{1}{e}\right)^{\frac{6}{2.5}} = 10(e^{-\frac{6}{2.5}}).$$

Set 2.5 on "C" opposite e^{-1} on "LLO2" (e^{-1} is the left index of "LLO2").
Opposite 6 on "C" read I = 0.091×10 amp= **0.91** *amp on "LLO3".*

B. Finding logarithms of numbers less than unity.

Since decile powers of e^{-1} on the "LLO1", "LLO2" and "LLO3" are placed opposite the indices of the "D" scale, the natural logarithms of given numbers on the "LLO1", "LLO2" and "LLO3" scales are read on the "D" scale opposite the given numbers. The value of the logarithm will be negative and the decimal point can be readily located by referring to the powers of e marked at the ends of "LLO" scales.

Example 6. Find ln 0.983 and ln 0.18.

Set hairline through 0.983 on "LLO1" and through 0.18 on "LLO3".
Under hairline on "D" read ln 0.983 = − **0.0172** *and ln 0.18 = −***1.72**.

For numbers less than unity common logarithms (to the base 10) are found by use of the "L" scale as explained in Section 46-B.

TO FIND LOGARITHMS TO ANY BASE other than e or 10, the slide rule is set as in determining the power of a given number, as explained in Section 46-B. In settings for which the results lie off the end of the "LLO" scale on which you start, keep in mind the idea of the chain of "LLO" scales pictured in Figure 56.

Example 7. Find $\log_{0.6} 0.25$.

Set right index of "C" to 0.6 on "LLO2".
Opposite 0.25 on "LLO2" read $\log_{0.6} 0.25$ = **2.71** *on "C".*

Example 8. Find $X = \log_2 0.95$.

$0.95 = 2^X = (\frac{1}{2})^{-X}$.
Set right index of "C" opposite 0.50 ($=\frac{1}{2}$) on "LLO2".
*Opposite 0.95 on "LLO1" read X = −***0.074** *on "C".*

Example 9. Find $\log_{0.98} 0.0032$.

Set left index of "C" opposite 0.98 on "LLO1"
Opposite 0.0032 on "LLO3" read $\log_{0.98} 0.0032$ = **284** *on "C".*

Exercises

Evaluate the following exercises. If there is an unknown letter value given, determine this unknown. (See Exercise 10).

1. $e^{-3.6}$	6. $\sqrt[7]{0.0108}$	11. $e^{-X} = 0.564$	15. ln 0.1
2. $(0.895)^{4.56}$	7. $(0.018)^{0.6}$	12. $e^{-X} = 0.97$	16. $\log_{0.5} 0.01$
3. $e^{-0.012}$	8. $(0.018)^{0.06}$	13. $(X)^{0.67} = 0.954$	17. $\log_{\pi} 0.92$
4. $(0.563)^{0.97}$	9. $(0.018)^{0.006}$	14. $(X)^{1.50} = 0.67$	18. log 0.0212
5. $\sqrt[5]{0.735}$	10. $e^{-5.67} = X$		

Answers to the above exercises:

1. 0.0273	8. 0.786	
2. 0.603	9. 0.9762	15. − 2.30
3. 0.9881	10. X = 0.0035	16. 6.63
4. 0.573	11. X = − 0.573	17. − 0.0728
5. 0.9402	12. X = − 0.0305	18. 8.326 − 10
6. 0.523	13. X = 0.932	
7. 0.09	14. X = 0.766	

48. Readings Beyond the Limits of the "LL" Scales.

If, in calculations involving powers and logarithms, one has to deal with numbers greater than 22,026 (maximum number on "LL3") or less than 0.00005 (minimum number on "LLO3") then one of the following methods may be resorted to:

Method 1. By factoring (splitting) the base, as $28^5 = (4 \times 7)^5 = 4^5 \times 7^5$. When solved in the usual way, $4^5 = 1024$ and $7^5 = 16{,}807$. Multiplying these results together, we obtain 17,210,368.

Method 2. By breaking (splitting) the exponent, as $28^5 = 28^2 \times 28^3$. When solved in the usual way, $28^2 = 784$ and $28^3 = 21{,}952$. Multiplying these results together, we obtain 28^5 equals 17,210,368.

Method 3. By means of common logarithms, using the L Scale, log 28 equals 1.44716, multiplied by 5 = 7.23580. The number whose log is 7.23580, we find to be 17,210,368.

Example 1. Evaluate $(128)^4$.

$(128)^4 = (1.28)^4 \times (100)^4$.

Set left index of "C" to 1.28 on "LL2".
Opposite 4 on "C" read 2.68 on "LL2".
Hence, $(128)^4 = \mathbf{2.68} \times \mathbf{10^8}$.

Example 2. Evaluate $e^{23.2}$ by splitting the exponent $e^{23.2} = e^{10+10+3.2} = e^{10} \times e^{10} \times e^{3.2}$.

$e^{10} = 22{,}026$.

Set hairline to 3.2 on "D".
Under hairline on "LL3" read $e^{3.2} = 24.5$.
Set right index of "C" to 22, 026 on "D".
Opposite 24.5 on B-RIGHT read $e^{23.2} = \mathbf{11.85} \times \mathbf{10^9}$ *on "A".*

Example 3. Evaluate $(0.0129)^{5.2}$ by factoring the base.

$(0.0129)^{5.2} = (1.29)^{5.2} \times (10^{-2})^{5.2} = (1.29)^{5.2} \times (10^{-2})^5 \times (10^{-2})^{0.2}$.

Set right index of "C" opposite 1.29 on "LL2".
Opposite 5.2 on "C" read $(1.29)^{5.2} = 3.76$ *on "LL3".*
Set left index of "C" opposite 0.01 (= 10^{-2}*) on "LLO3".*

Opposite 0.2 on "C" read $10^{-0.4} =$ *0.398 on "LLO3".*
Set left index of "C" to 3.76 on "D".
Opposite 0.398 on "CF" read 1.494 on "DF".
Hence, $(0.0129)^{5.2} = \mathbf{1.494} \times \mathbf{10^{-10}}$.

Example 4. Evaluate $(0.0129)^{5.2}$ by use of logarithms.

Set hairline to 1.29 on "D".
Under hairline on "L" read log (1.29) = 0.1104.
Hence, log (0.0129) = −2 + 0.1104.
Set left index of "C" to 0.1104 on "D".
Opposite 5.2 on "C" read 0.574 on "D".
Hence, log $(0.0129)^{5.2} = -10.4 + 0.574$.
$= -10 + 0.174$.
Set hairline to 0.174 on "L".
Under hairline on "D" read 1.494.
Therefore, $(0.0129)^{5.2} = \mathbf{1.494} \times \mathbf{10^{-10}}$.

If powers of numbers very near 1.00 are needed, then the "LL" scales may also prove inadequate, since the lowest value on "LL1" is 1.01 and the highest value on "LLO1" is 0.9905. For such calculations one may make use of the binominal expansion.

$$(1 \pm X)^n = 1 \pm nX + \frac{n(n-1)}{2}X^2 \pm \ldots .$$

If nX is less than 0.05, then the first two terms in the series give the correct value with an error of about 1 part in 1,000, or less.

$$(1 \pm X)^n = 1 \pm nX \text{ (approximately)}$$

If nX is larger than 0.05, then three terms in the series may be used, or another method of calculation may be tried.

Example 5. Evaluate $(1.0032)^{4.32}$.

$(1 + 0.0032)^{4.32} = 1 + 4.32\ (0.0032)$.

Set left index of "C" to 0.0032 on "D".
Opposite 4.32 on "CF" read 0.0138 on "DF".
Hence $(1.0032)^{4.32} = \mathbf{1.0138}$.

Example 6. Evaluate $(0.9995)^{0.47}$.

$(1 - 0.0005)^{0.47} = 1 - 0.47\ (0.0005)$
$= 1 - 0.000235 = \mathbf{0.999765}$

Exercises

Evaluate the following (using all methods applicable).

1. $(24)^6$
2. $(128)^{4.21} = (128)\ (128)^{1.21}\ (128)^2$
3. $\left[\dfrac{21 \times 8.2}{3.21}\right]^4$
4. $\dfrac{(52 \times 8.134)^3}{\sqrt{42}}$
5. $\left[\dfrac{\sqrt[3]{8.18}\ (51.2)}{\sqrt{6.92}}\right]^4$
6. $(1.0039)^{5.21}$
7. $(1.0085)^{0.398}$
8. $(1.000069)^{9.2}$
9. $\left[(127)\ \dfrac{(0.000124)}{(0.01564\ )}\right]^{0.05}$
10. $[1.00072]^{9.35/6.12}$

Answers to above exercises.

1. 1.91×10^8
2. 7.46×10^8
3. 8.28×10^6
4. 11.64×10^6
5. 2.35×10^6
6. 1.0203
7. 1.00338
8. 1.000635
9. 1.00025
10. 1.0011

49. Theory Underlying Construction of the "LL" Scales.

In the preceding four sections the use of the "LL" scales has been explained in detail, but little has been said regarding their construction. Let us see how the "LL" scales have been laid out to have such useful properties.

In Figure 55 it can be seen that the numbers located at the ends of the "LL" scales are decile powers of e and that the opposed numbers on the "D" scales are simply the exponents of e. Since these exponents of e are, by definition, the natural logarithms of the numbers marked on the "LL" scales, it follows that the numbers marked on the "D" scale are, assuming proper location of their decimal points, the natural logarithms of the opposed numbers marked on the "LL" scale. The relationship is true throughout the length of the scales because of the following facts:

1. The distance from the left index to a given number n marked on the "D" scale is equal to the mantissa of the log n multiplied by a scale factor of 25 centimeters. For example, the mark for 3.2 is located at a distance (log 3.2) × 25 cm. = 0.505 × 25 cm. = 12.6 cm. = 4.97 inches from the left index. Measure it!

2. The distance from the left index to a given number X on an "LL" scale is equal to the mantissa of log (ln X) multiplied by a scale factor of 25 centimeters. (Since the natural logarithm of X is itself simply a number, we can take its common logarithm just as we could the common logarithm of any other number.) For example, the mark for 24.5 on "LL3" is located at a distance [log (ln 24.5)] × 25 cm. = (log 3.2) × 25 cm. = 0.505 × 25 cm. = 12.6 cm. = 4.97 inches from the left index. Measure it! (It is because distances on "LL" scales are proportional to log (ln X) that the scales are called Log Log scales.)

3. If a number n on the "D" scale and a number X on an "LL" scale are the same distance from the left index, it follows that
   $\log(\ln X) \times 25 \text{ cm.} = \log n \times 25 \text{ cm.}$
   $\log(\ln X) = \log n,$
   $\ln X = n,$ and
   $X = e^n.$

   Since n = 3.2 on "D" is the same distance from the left index as is Y = 24.5 on "LL3", it follows that ln 24.5 = 3.2 or that $24.5 = e^{3.2}$. Correct location of the decimal point in n is assumed.

Now let us see how this layout of the "LL" scales permits the easy determination of powers of numbers as described in the preceding sections. Look back at Figure 54. The distance from the left index of "LL2" to the mark for 1.23 is equal to log (ln 1.23) × 25 cm. The left index of the "C" scale was set to 1.23 on "LL2" and a distance equal to (log 1.84) × 25 cm. was added to the distance from the left index. If we call Y the number on "LL2" opposite 1.84 on "C", then the equation for its distance from the left index of "LL2" is:

$$\log(\ln Y) \times 25 \text{ cm.} = \log(\ln 1.23) \times 25 \text{ cm.} + \log 1.84 \times 25 \text{ cm.}$$

Dividing by 25 cm. and applying the law for addition of logarithms to the right side, we have:

$$\log(\ln Y) = \log[1.84\ (\ln 1.23)], \text{ or}$$
$$\ln Y = 1.84\ (\ln 1.23) = \ln(1.23)^{1.84}.$$

Hence, $Y = 1.23^{1.84}$. On the "LL2" scale we read Y = 1.464.

In general, if $Y = X^n$ then
$\ln Y = n \ln X$, and
$\log (\ln Y) = \log (\ln X) + \log n$.

Add the *n*-distance on "C" to the X-distance on an "LL" scale to obtain the Y-distance on an "LL" scale.

The "LLO1", "LLO2" and "LLO3" scales differ only in that the distance from the left index to a number X is proportional to the common logarithm of the power of e^{-1} which would yield X. Due to the improved design of these rules, the scale factor is the same for these scales, as for the "LL1", "LL2" and "LL3" scales; namely 25 cm. This permits the "LLO" scales to be read in conjunction with the "C" and "D" scales in the same manner as the "LL" scales. For instance, $0.135 = e^{-2} = (e^{-1})^2$. Hence, the distance from the left index of "D" to the mark for 0.135 on "LLO3" is equal to (log Z) x 25 cm. = 0.301 x 25 = 7.525 cm. = 2.96 inches. Hence, if $Y = X^n$, then

$$\log_{e^{-1}} Y = n \log_{e^{-1}} X, \text{ and}$$

$$\log \left(\log_{e^{-1}} Y\right) = \log \left(\log_{e^{-1}} X\right) + \log n.$$

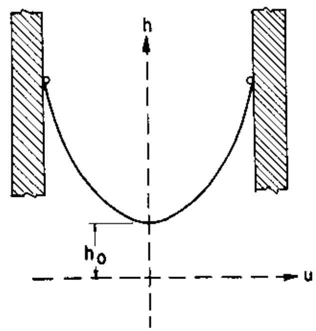
Multiply each term by 25 cm. The terms are then converted into distance on the "LLO1", "LLO2", "LLO3" and "D" scales. Therefore, if $Y = X^n$, add the n – distance on "D" to the X – distance on the proper "LLO" scale to yield the Y – distance on the proper "LLO" scale.

CHAPTER IX.

HYPERBOLIC FUNCTIONS

50. Introduction to Hyperbolic Functions.

RELAX! Don't be afraid of hyperbolic functions. If you suspend a uniform flexible cable from its ends and let it hang while supporting only its own weight, the curve it assumes will look like that in Figure 57. The curve is not an arc of a circle; it is not a parabola. To describe the shape of this curve we might fit in *u* and *h* axes as shown and make a table of values of *h* distances corresponding to given values for *u* distances. For one weight of cable and for h_0 equal to 1 foot, the table would be as given at the right in Figure 57.



u (ft.)	h (ft.)
0	1.000
±0.5	1.128
±1.0	1.543
±1.5	2.352
±2.0	3.762
etc.	etc.

Fig. 57

Just as a proper table of corresponding values for *y* and *x* will define the function *y* = cos x, so the above table of values, if extended, would define the function h = cosh u. By cosh u we mean a numerical value found in the *h* column of the above table opposite the given value of *u*. Thus, cosh 2 = 3.762 and cosh 0 = 1.000. A more extended table would give values of cosh u for values of *u* between the values in the simple table above.

Other tables of values define the other hyperbolic functions: sinh u, tanh u, sech u, csch u, and coth u. These names are read "hyperbolic sine of u", "hyperbolic tangent of u", etc.

Graphs of the three principal hyperbolic functions are shown in Figure 58 for a small range of values for u. The remaining hyperbolic functions are defined to be the reciprocals of the principal functions. Thus, coth u = 1/tanh u, sech u = 1/cosh u, csch u = 1/sinh u.

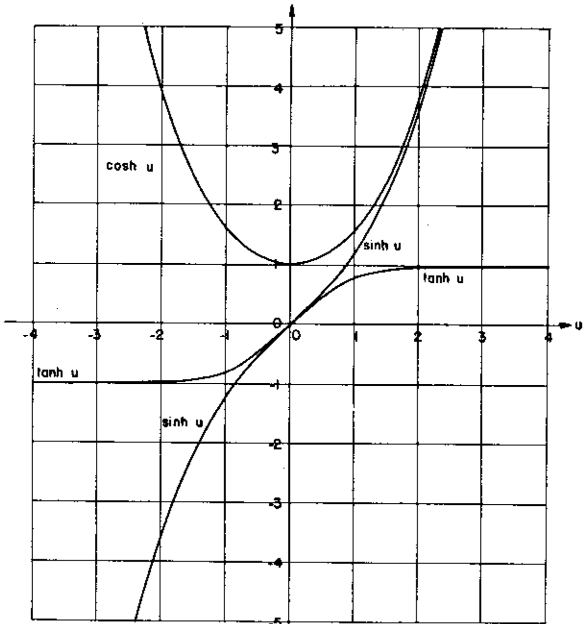


Fig. 58

The hyperbolic functions receive the name "hyperbolic" from the relations they bear to a rectangular hyperbola and the analogy between these relations and those borne by the plane trigonometric

functions to a circle. The equation for the rectangular hyperbola in Figure 59 is:

$$x^2 - y^2 = a^2.$$

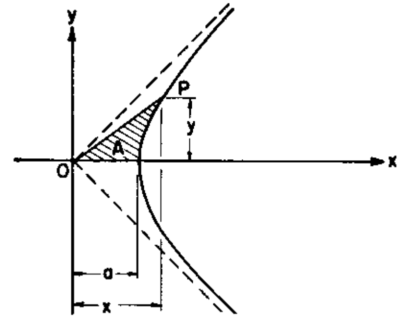


Fig. 59

For any point P with coordinates x and y the hyperbolic functions are defined as follows:

sinh u = y/a	csch u = a/y
cosh u = x/a	sech u = a/x
tanh u = y/x	coth u = x/y

As the point P moves from the vertex out along the hyperbola, the values of the hyperbolic functions vary with the values of u in the manners indicated by their graphs in Figure 58.

The hyperbolic angle u is defined geometrically in terms of the shaded area A and the vertex distance a:

$$u = \frac{2A}{a^2}.$$

This relation is analogous to that for a circle of radius a, in which a sector with central angle θ has an area given by $A = \frac{1}{2}a^2\theta$. From this equation we find $\theta = \frac{2A}{a^2}$. For both the circle and the hyperbola the areas called A are directly proportional to the angles θ and u. But the angle u cannot be interpreted geometrically as the angle between the positive x-axis and the radius line OP. This dissimilarity arises from the fact that the radius line OP is not of constant length, so that equal increments in the angle from the x-axis to OP do not add equal increments to the area A. Consequently, we just have to think of the angle u as a quantity proportional to A and defined by the equation $u = \frac{2A}{a^2}$.

The hyperbolic functions may also be defined in terms of exponentials to the base *e*. (The number e = 2.7182818 . . . is the base of natural logarithms and is the number at the left index of the "LL3" scale and the right index of the "LL2" scale.)

$$\sinh u = \tfrac{1}{2}(e^{u} - e^{-u})$$
$$\cosh u = \tfrac{1}{2}(e^{u} + e^{-u})$$
$$\tanh u = (e^{u} - e^{-u})/(e^{u} + e^{-u})$$

Certain fundamental identities exist between hyperbolic functions, analogous to the relations between plane trigonometric functions. These identities, true for any number u, may be proved from the geometric definitions or from the definitions in terms of exponentials.

$$\tanh u = \sinh u/\cosh u$$
$$\cosh^2 u - \sinh^2 u = 1$$
$$1 - \tanh^2 u = \operatorname{sech}^2 u$$

Formulae for hyperbolic functions of the sum and difference of two numbers and for double and half angles are also analogous to similar trigonometric functions, though some signs differ.

$$\sinh (u \pm v) = \sinh u \cosh v \pm \cosh u \sinh v$$
$$\cosh (u \pm v) = \cosh u \cosh v \pm \sinh u \sinh v$$
$$\tanh (u \pm v) = (\tanh u \pm \tanh v)/(1 \pm \tanh u \tanh v)$$
$$\sinh 2u = 2 \sinh u \cosh u$$
$$\cosh 2u = \cosh^2 u - \sinh^2 u$$
$$\sinh u/2 = \sqrt{\tfrac{1}{2}(\cosh u - 1)}$$
$$\cosh u/2 = \sqrt{\tfrac{1}{2}(\cosh u + 1)}$$

Since the hyperbolic functions are not periodic, we do not have formulae comparable to those in plane trigonometry for reducing a given function of any angle to a function of an angle less than 90°. But we do have relations between the hyperbolic functions for equal positive and negative values of u.

$$\sinh (-u) = -\sinh u$$
$$\cosh (-u) = \cosh u$$
$$\tanh (-u) = -\tanh u$$

These relations can be checked and remembered readily by reference to the graphs of the functions in Figure 58.

51. The Hyperbolic Sine and Tangent Scales.

In physical phenomena we sometimes find one physical quantity varying as a hyperbolic function of another physical quantity. Such functions appear frequently enough in scientific and engineering work, particularly in electrical engineering, so that it becomes desirable to have a slide rule with scales representing hyperbolic functions.

The DIETZGEN MANIPHASE MULTIPLEX VECTOR TYPE LOG LOG RULE has scales representing hyperbolic sines and tangents, which are the only scales needed for making calculations involving hyperbolic functions. The "Sh1" and "Sh2" scales, which are essentially two sections of one long scale, represent values of sinh u for values of u from 0.1 to 3.0, and the "Th" scale represents values of tanh u for values of u from 0.1 to ∞.

A. The hyperbolic tangent scale.

As is usual in the design of slide rule scales, the distance from the left hand index is made proportional to the mantissa of the logarithm of the quantity for which the scale is constructed. Thus, on the "Th" scale, distances from the left index are equal to the mantissa of log (tanh u) times a scale factor of 25 centimeters. *But the graduations are marked with values of the angle u.* This is similar to the layout and marking of the "S", "T<45", "T>45", and "ST" scales for the trigonometric functions.

Since the "D" scale is laid out for numbers N and since distances from its left index are equal to the mantissa of log N times the same scale factor of 25 centimeters, therefore *the numbers N marked on the "D" scale are equal to the hyperbolic tangents of the angles u marked on the "Th" scale*, assuming proper location of the decimal point.

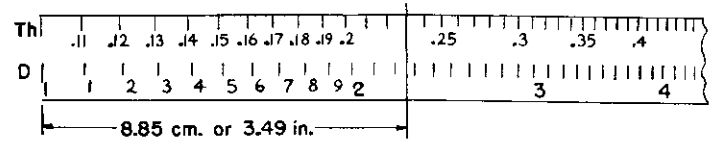


Fig. 60

In Figure 60 are shown portions of the "D" and "Th" scales as they relate to each other on the body of the rule. A hairline has been drawn through the value 0.23 on the "Th" scale. Since (from a table of hyperbolic functions) tanh 0.23 = 0.226 and since the mantissa of log 0.226 = 0.3541, therefore the distance from the left index to the hairline is 0.354 × 25 cm. = 8.85 cm., or 3.49 in. Measure the distance with a ruler to check this relationship. Opposite the hairline on the "D" scale we can read the number 2.26. This, except for proper location of the decimal point, is the value of the hyperbolic tangent of the number 0.23 under the hairline on the "Th" scale. To determine the decimal point we may note, from the graph of tanh u in Figure 58, that values of tanh u always lie between 0 and 1.0. *For the range of u-values shown on the slide rule (0.1 to ∞), the values for tanh u lie in the range from 0.1 to 1.00,* and the decimal point is readily located. From the setting of the hairline in Figure 60 we read tanh 0.23 = 0.226. *For values of u smaller than 0.1, tanh u may be taken equal to u.* The error in this approximation is less than 3 parts in 1000. For instance, when u = 0.0800, then tanh u = 0.0798.

Note that all values of *u* from 3 to ∞ are squeezed into the small length of the "Th" scale between the ruling marked "3" and the ruling at the right index of the "Th" scale. For most practical purposes tanh u may be taken equal to 1.00 for values of *u* greater than 3.

Example 1. What is the value of tanh (−1.23)?
Set hairline to 1.23 on "Th" scale.
Under hairline on "D" scale read 0.843.
Since tanh (−u) = −tanh u, tanh (−1.23) = **−0.843.**

Example 2. What is the value of tanh 0.071?
For such small values of u, tanh u *is very nearly equal to* u.
So tanh 0.071 = **0.071**, *very nearly.*

Since it is possible, given *u*, to find tanh u by use of the "Th" and "D" scales, it is also possible to find *u* if tanh u is given.

Example 3. What number *u* has its hyperbolic tangent equal to 0.800?
Set hairline to 0.800 on "D" scale.
Under hairline on "Th" scale read u = **1.098.**

Exercises

1. Find the hyperbolic tangents of the following numbers.

a. 0.125	d. 14.62
b. 0.0236	e. 1.04
c. −0.792	f. −3.20

2. Find the numbers which have hyperbolic tangents equal to the following.

a. 0.828	d. 0.945
b. 0.0925	e. −0.0631
c. −0.276	f. 0.106

Answers to the above exercises.

1.	a. 0.1244	d. 1.00
	b. 0.0236	e. 0.778
	c. −0.660	f. −0.997
2.	a. 1.18	d. 1.78
	b. 0.0925	e. −0.0631
	c. −0.283	f. 0.1064

B. The hyperbolic sine scales.

The values of sinh u vary from 0 to ∞ as *u* varies from 0 to ∞. Look at the graph for sinh u in Figure 58 again. *For values of u less than 0.1, the values of sinh u may be taken equal to u.* The error in this approximation is less than 2 parts in 1000. For instance, when u = 0.0860, then sinh u = 0.0861.

To represent the wide range of values for sinh u, two scales, "Sh1" and "Sh2", are placed on the slide rule. The distance from the left index to a number *u* on either scale is equal to the mantissa of log (sinh u) multiplied by a scale factor of 25 centimeters. *The graduations are marked with values of u,* not with the values of sinh u. Because the "D" scale is laid out similarly and with the same scale factor, therefore *the numbers N marked on the "D" scale are equal to the hyperbolic sines of numbers marked on the "Sh1" and "Sh2" scales.*

The "Sh1" scale covers values of *u* between 0.100 to 0.881. For these values of *u* the values of sinh u as read on the "D" scale range from 0.100 to 1.00.

The "Sh2" scale, covering values of *u* from 0.881 to 3.00, would extend to the right from the "Sh1" scale and would be read against an added section of the "D" scale, if such existed. Since it does not, the "Sh2" scale has been brought back under the "D" scale, and values of sinh u are read on "D". The decimal point may be located properly by noting that as *u* varies from 0.881 to 3.00 along the "Sh2" scale, the values of sinh u vary from 1.00 to 10.0 along the "D" scale.

Example 4. What is the value of sinh 0.268?

Set hairline to 0.268 on "Sh1".
Under hairline on "D" read **0.271.**

Example 5. What is the value of sinh 0.0255?

For such small values of u, *sinh u is very nearly equal to* u.
So sinh 0.0255 = **0.0255.**

Example 6. What is the value of sinh (−1.92)?

Set hairline to 1.92 on "Sh2".
Under hairline on "D" read 3.34.
Since sinh (−u) = −sinh u, therefore sinh (−1.92) = **−3.34.**

Since it is possible, given *u*, to find sinh u by use of the "Sh1" or "Sh2" and "D" scales, it is also possible to find *u* if sinh u is given.

Example 7. What number *u* has its hyperbolic sine equal to 1.16?

Set hairline to 1.16 on "D".
Under hairline on "Sh2" read u = **0.990.**

What about values for sinh u when *u* is greater than 3? When *u* is greater than 3, we make use of the algebraic definition of the hyperbolic sine.

$$\sinh u = \tfrac{1}{2}(e^{u} - e^{-u})$$

Example 8. Find sinh 4.

Sinh 4 = $\tfrac{1}{2}(e^{4} - e^{-4}) = \tfrac{1}{2}(e^{4} - 1/e^{4})$.
First find e^{4}, *noting that e = 2.71828 . . . is the number at the left end of the "LL3" scale.*
Set hairline to 4 on "D" scale.
Under hairline on "LL3" scale read $e^{4} = 54.6$.
Next find $1/e^{4} = 1/54.6$.
Set hairline to 54.6 on "C" scale.
Under hairline on "CI" scale read $1/e^{4} = 0.0190$.
Hence, sinh 4 = $\tfrac{1}{2}(54.6 - 0.0190)$
$= \tfrac{1}{2}(54.58) =$ **27.29.**

In the above example the second term in the parentheses was very small, and we could, with a high degree of accuracy, have taken sinh u equal to $\tfrac{1}{2}e^{u}$. Indeed, *for all values of u greater than 3, we may take sinh u equal to* $\tfrac{1}{2}e^{u}$, with an error from the true value of less than 3 parts in 1000.

For the rare instances in which *u* is greater than 10, we shall not be able to read the value of e^{u} on the "LL3" scale. The problem may then be solved by a resort to logarithms.

Example 9. Find sinh 22.1.

Sinh 22.1 equals very nearly $\tfrac{1}{2}e^{22.1}$.
Log $(\tfrac{1}{2}e^{22.1})$ *= 22.1 (log e) − log 2.*
Set hairline to e = 2.71828 . . . on "D".
Under hairline on "L" read log e = 0.4343.
Set left index of "C" to 22.1 on "D".
Opposite 0.4343 on "C" read 22.1 (log e) = 9.59 on "D".
Set hairline to 2.00 on "D".
Under hairline on "L" read log 2 = 0.301.
Log $(\tfrac{1}{2}e^{22.1})$ *= 9.59 − 0.301 = 9.29.*
Set hairline to 0.29 on "L".
Under hairline on "D" read 1.95.
Since the characteristic of log $(\tfrac{1}{2}e^{22.1})$ *is 9, therefore* $\tfrac{1}{2}e^{22.1} = 1.95 \times 10^{9}$ *or 1,950,000,000.*
Hence, sinh 22.1 = $\mathbf{1.95 \times 10^{9}}$.

Exercises

1. Find the hyperbolic sines of the following numbers.

a. 0.125	d. 14.6
b. 0.0236	e. 1.04
c. −0.792	f. −3.20

2. Find the numbers which have hyperbolic sines equal to the following.

a. 0.828	d. 9.45
b. 0.0925	e. 16.3
c. −2.76	f. 0.106

Answers to the above exercises.

1.	a. 0.1254	d. 1,100,00
	b. 0.0236	e. 1.238
	c. −0.878	f. −12.27
2.	a. 0.754	d. 2.94
	b. 0.0925	e. 3.48
	c. 1.740	f. 0.1058

52. Determining Other Hyperbolic Functions.

A. *Finding hyperbolic cosines.*

For numbers u less than 0.100, the values of cosh u may be taken equal to 1.00, with an error of less than 5 parts in 1000. Look back at Figure 58, and you will see how reasonable this approximation is.

For numbers u which appear on the "Sh1" and "Sh2" scales, i. e. for numbers between 0.100 and 3.00, values of cosh u can be found on the slide rule. The basic formula involved is

$$\tanh u = \sinh u/\cosh u, \text{ or}$$
$$\tanh u \cosh u = \sinh u.$$

Interpreted in terms of distances along the slide rule scales, the second equation above becomes

$$\log(\tanh u) + \log(\cosh u) = \log(\sinh u).$$

This means that a distance corresponding to cosh u must be added to the tanh u distance to give a distance corresponding to sinh u. In Figure 61 the distance corresponding to cosh 0.7 is indicated.

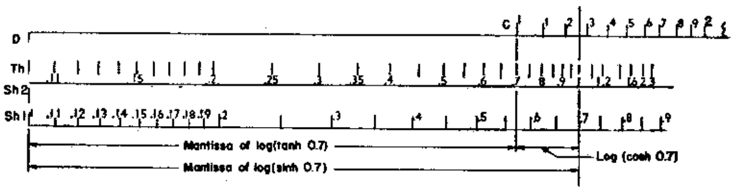


Fig. 61

To find the value of cosh 0.7, evaluate the distance corresponding to cosh 0.7 by using the "C" scale.

Example 1. Find the value of cosh 0.7.
Set left index of "C" to 0.7 on "Th".
Opposite 0.7 on "Sh1" read cosh 0.7 = **1.255** *on "C".*

The slide may be pulled out and turned over so that the "C" scale is brought onto the same side of the rule as that on which the hyperbolic scales are located. The decimal point may be placed correctly by noting that cosh u varies between 1.00 and 10.0 for values of u on the slide rule.

To determine values of cosh u for values of u between 0.881 and 3.00, one must use the "Sh2" scale in conjunction with the "Th" scale. The setting of the rule for such a determination is shown in Figure 62. The value of cosh 1.10 is desired.

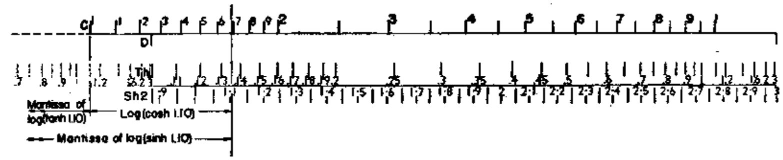


Fig. 62

The "Sh2" scale should extend to the right from the right index of the "Th" scale. This effect has been obtained in Figure 62 by considering an imaginary "Th" scale translated to the left one scale length. When the right index of the "C" scale is placed opposite 1.10 on the true "Th" scale, the left index of the "C" scale is brought opposite 1.10 on the imaginary "Th" scale. So the "C" scale is in position to evaluate cosh 1.10 by reference to the imaginary "Th" and the true "Sh2" scales.

Example 2. Find the value of cosh 1.10.
Set the right index of "C" opposite 1.10 on "Th".
Opposite 1.10 on "Sh2" read cosh 1.10 = **1.67** *on "C".*

If u is greater than 3.00, the value of cosh u cannot be found from the hyperbolic scales. For such cases, we make use again of the algebraic definition.

$$\cosh u = \tfrac{1}{2}(e^u + e^{-u})$$

When u is greater than 3.00, cosh u may be taken equal to $\frac{1}{2}e^u$, with an error of less than 3 parts in 1000. Since this approximation is the same as that used for sinh u when u was greater than 3.00, we are saying that cosh u and sinh u are very nearly equal for values of u greater than 3.00 and that both may be taken equal to $\frac{1}{2}e^u$. Methods for calculating $\frac{1}{2}e^u$ were given in Section 51.

Since it is possible, given u, to find cosh u by use of the slide rule scales, it is also possible to find u if cosh u is given. The reverse of the process for finding cosh u is laborious, involves several approximations in setting of the slide. So it is suggested that sinh u be determined from the given value of cosh u and that u be determined from the value of sinh u.

$$\sinh u = \sqrt{\cosh^2 u - 1}$$

Example 3. What number u has its hyperbolic cosine equal to 1.50?

Sinh $u = \sqrt{(1.50)^2 - 1} = \sqrt{2.25 - 1}$
$= \sqrt{1.25}.$
Set hairline to 1.25 on "A-LEFT".
Under hairline on "Sh2" read $u =$ **0.962.**
(For $u = -0.962$, *cosh* $u = 1.50$ *also.)*

B. *Finding hyperbolic cotangents, secants, and cosecants.*

Since the remaining hyperbolic functions are merely the reciprocals of those described above, we can evaluate the former by setting the slide rule to read the reciprocals of the values determined for tanh u, cosh u, and sinh u.

Thus, coth u = 1/tanh u, and values of the hyperbolic cotangent are readily found by use of the "Th" and the "DI" scales. Since values of tanh u on the rule vary from 0.100 up to 1.00, values of coth u will vary from 10.0 down to 1.00.

Example 4. Find the value of coth 0.390.

Set hairline to 0.390 on "Th".
Under hairline on "DI" read coth 0.390 = **2.69.**

Since csch u = 1/sinh u, values of the hyperbolic cosecant are readily found by use of the "Sh1" and "Sh2" scales in conjunction with the "DI" scale. Values of sinh u vary from 0.100 up to 1.00 along the "Sh1" scale; so values of csch u will vary from 10.0 down to 1.00 when u appears on the "Sh1" scale. When u appears on the "Sh2" scale, values of csch u will lie between 1.00 and 0.100.

Example 5. Find the value of csch 1.55.

Set hairline to 1.55 on "Sh2".
Under hairline on "DI" read csch 1.55 = **0.445.**

Finally, since sech u = 1/cosh u, we may find values of sech u by setting the rule as if to determine cosh u and then reading the answer on the "CI" scale rather than on the "C" scale. Values of cosh u obtainable on the slide rule lie between 1.00 and 10.0; so the values of sech u will lie between 1.00 and 0.100.

Example 6. Find the value of sech 0.538.

Set left index of "C" to 0.538 on "Th".
Opposite 0.538 on "Sh1" read sech 0.538 = **0.871** *on "CI".*

When coth u, sech u, or csch u is desired for a number u which does not appear on the slide rule, it may be found by determining the value of the reciprocal function and then inverting the result.

Exercises

1. Find the hyperbolic cosines of the following numbers.

a. 0.125	d. −0.792
b. 0.0236	e. 14.6
c. 1.04	f. −3.20

2. Find the numbers which have hyperbolic cosines equal to the following.

a. 1.828	c. 2.76
b. 1.093	d. 15

3. Find the values of the following hyperbolic functions.

a. coth 0.872	d. sech 1.06
b. sech 0.235	e. csch 2.94
c. csch 0.153	f. coth 0.0235

Answers to the above exercises.

1.	a. 1.008	d. 1.33
	b. 1.00	e. 1,100,000
	c. 1.591	f. 12.27
2.	a. 1.211	c. 1.674
	b. 0.428	d. 3.40
3.	a. 1.424	d. 0.619
	b. 0.973	e. 1.060
	c. 0.651	f. 42.6

53. Summary and Applications of Hyperbolic Functions of Real Numbers.

Finding values of sinh u.

Value of u between	*Value of sinh u between*	*How found?*
0 & 0.0998	0 & 0.1000	Take sinh u equal to *u*.
0.0998 & 0.881	0.1000 & 1.000	Opposite *u* on "Sh1" find value of sinh u on "D".
0.881 & 2.998	1.000 & 10.00	Opposite *u* on "Sh2" find value of sinh u on "D".
2.998 & ∞	10.00 & ∞	Take sinh u equal to $\frac{1}{2}e^u$.

If *u* is negative, sinh u will be negative.

Finding values of tanh u.

Value of u between	*Value of tanh u between*	*How found?*
0 & 0.1003	0 & 0.1000	Take tanh u equal to *u*.
0.1003 & 3.00	0.1000 & 0.995	Opposite *u* on "Th" find value of tanh u on "D".
3.00 & ∞	0.995 & 1.000	Take tanh u = 1.00.

If *u* is negative, tanh u will be negative.

Finding values of cosh u.

Value of u between	*Value of cosh u between*	*How found?*
0 & 0.1003	1.000 & 1.005	Take cosh u = 1.00.
0.1003 & 0.881	1.005 & 1.414	Set left index of slide to *u* on "Th". Opposite *u* on "Sh1" read value of cosh u on "C".
0.881 & 2.993	1.414 & 10.00	Set right index of slide to *u* on "Th". Opposite *u* on "Sh2" read value of cosh u on "C".
2.993 & ∞	10.00 & ∞	Take cosh u = $\frac{1}{2}e^u$.

If *u* is negative, cosh u will nevertheless be positive.

Finding values of other hyperbolic functions.

Find the value of the reciprocal function by the correct method given above and take the inverse of that value.

coth u = 1/tanh u
sech u = 1/cosh u
csch u = 1/sinh u

The operation is quickly performed for a value of *u* on the slide rule by adjusting the rule for determining the value of the reciprocal function on the "C" scale and by reading the inverse on the "CI" scale.

Applications.

Since all values of hyperbolic functions as determined on the slide rule are found on the "C" and "D" scales, those values can be multiplied, divided, squared, and treated otherwise in calculations exactly as could any number on the "C" and "D" scales.

Example 1. A certain viscous medium exerts a drag proportional to the square of the velocity of a body moving through it. In such a medium the velocity (v) of a falling body will vary with time of falling (t) in accordance with the following formula:

$$v = V \tanh (gt/V),$$

where V is the maximum velocity attained, and g = 32.2 ft./sec. is the gravitational acceleration at the earth's surface.

A certain falling body has a maximum velocity of 21.5 ft./sec. Find its velocity after 0.75 sec. of falling.

Solution:

u = gt/V = (32.2) (0.75)/21.5.
Set 21.5 on "C" to 32.2 on "D".
Opposite 0.75 on "CF" read u = 1.123 on "DF".
v = V tanh u = 21.5 tanh 1.123.
Set right index of "C" to 1.123 on "Th".
Opposite 21.5 on "C" read v = **17.4 ft./sec.** *on "D".*

Example 2. A telephone line has resistance r = 12 ohms per mile and conductance g = 0.8×10^{-6} mhos per mile. The d-c. resistance at one end of the line with the other end short-circuited will be

$$R = \sqrt{\frac{r}{g}} \tanh \sqrt{rg}\, L,$$

where L is the length of the line in miles. Find R if L = 150 miles.

Solution:

Find $\sqrt{rg}\,L = \sqrt{(12)\,(0.8 \times 10^{-6})}\;150.$
Set right index of "B" to 12 on A-RIGHT.
Set hairline to 0.80 on B-RIGHT.
Bring right index of "C" to hairline.
Opposite 150 on "CF" read $\sqrt{rg}\,L = 0.465$ *on "DF".*
Set hairline to 0.465 on "Th".
Bring 0.80 on B-RIGHT to the hairline.
Opposite 12 on B-RIGHT read R = 1,680 ohms.

Example 3. When standing waves of sound are set up in a tube, the ratio of sound pressures at adjacent maxima and minima are given by

$$P\ max. / P\ min. = \coth \pi \propto .$$

The ratio P max./P min. is found by measurement to be 1.37. Find the value of $\propto$.

Solution:

Set hairline to 1.37 on "DI".
Under hairline on "Th" read $\pi \propto = 0.928.$
Set hairline to 0.928 on "DF".
Under hairline on "D" read $\propto = 0.2955.$

Exercises

1. Prove that $\cosh^2 u - \sinh^2 u = 1$ when u = 1.13.

2. A kite string hanging under its own weight forms a curve described by $h = 92 \cosh\left(\frac{x}{92}\right) - 88$, where *h* is the height of the kite in feet and *x* is the horizontal distance (in feet) from the person holding the string out to a spot vertically under the kite. If x = 225 ft., find *h*.

3. A telephone line with resistance r = 16 ohms per mile and conductance $g = 1.42 \times 10^{-6}$ mhos per mile is short-circuited 67 miles from the battery end of the line. The current I in amperes at the short-circuit is given by the formula

$$I = E\sqrt{g/r}/\sinh\sqrt{rg}\,L,$$

where L = the distance in miles from battery to the short-circuit, and E = 24 volts, the battery voltage. Find I.

4. In finding the capillary curve of the surface of a liquid near the vertical wall of a container it is necessary to find the number whose hyperbolic cosine is given. Find $u = \cosh^{-1}\frac{c}{h_o}$, if c = 0.550 and h = 0.293.

5. A centilever beam L inches long, weighing w pounds per inch, and subject to a horizontal pull P at its free end will have a maximum deflection D, in inches, given by

$$D = \frac{w}{Pk^2}\left[1 + \frac{K^2L^2}{2} - \operatorname{sech} kL - kL \tanh kL\right].$$

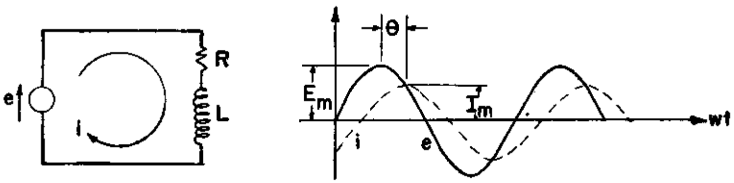
For a certain iron beam L = 144 inches, P = 6,000 lbs., w = 2.27 lbs. per inch, and k = 0.01225 per inch. Find D.

Answers to the above exercises.

1. $\cosh^2 1.13 - \sinh^2 1.13 = 2.92 - 1.92 = 1.00.$
2. h = 446 feet.
3. I = 0.0220 amperes.
4. u = 1.243.
5. D = 1.41 inches.

54. Introduction to Functions of Complex Numbers.

RELAX! Don't be afraid of functions of complex numbers. When a sinusoidal wave of voltage is impressed on an electric circuit consisting of resistance and inductance in series, the sinusoidal wave of current which pulses back and forth in the circuit may differ from the voltage wave in two attributes: (1) magnitude or maximum value of the wave, and (2) phase angle of the sinusoidal wave (or time at which the wave reaches its maximum point). This situation is pictured in Figure 63.



$e = E_m \sin wt$

$i = I_m \sin (wt - \theta)$

Fig. 63

The voltage source, perhaps the 110-volt wall outlet in your house, supplies a voltage which varies sinusoidally with time and has a maximum value E_m. The polarity of the source reverses sign twice every cycle, and the angular velocity w for the wave is 2π times the frequency of the voltage wave. Thus, the 60-cycle voltage wave in your house has $w = 2\pi\,(60) = 377$ radians per second. Such a voltage source will send an alternating sinusoidal current around the circuit. The current changes direction 120 times a second and has a maximum value in either direction of I_m. The maximum of the current wave occurs later in time than the maximum of the voltage wave as shown in Figure 63, where later instants of time are farther to the right. This time lag may be expressed in terms of the phase angle θ or in terms of the actual time lag $t_\theta = \frac{\theta}{w}$ in seconds.

Both the phase angle θ and the ratio between the magnitudes of E_m and I_m are determined by the values of resistance (R) and reactance ($X = wL$) in the circuit. Their effects on magnitude and phase angle of the current are expressed in terms of the *impedance* of the circuit whose magnitude Z equals the ratio of impressed voltage to current $\left(Z = \frac{E_m}{I_m}\right)$ and whose angle θ is the phase angle between the voltage and current waves. The impedance is related to resistance and reactance as indicated in the following equation and in Figure 64.

$$Z\underline{/\theta} = R + jX, \text{ where } j = \sqrt{-1}.$$

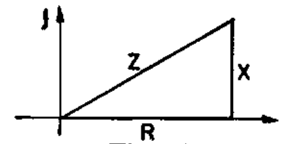


Fig. 64

The impedance is thus a complex number and may be graphed by laying off *R* along a horizontal axis and *X* parallel to the vertical j-axis.

When an alternating (sinusoidal) voltage source is connected to the end of a telephone line such as that described in Example 3 or in Exercise 3 of the preceding section, the current flowing in the line will be an alternating sinusoidal wave. The current may differ from the voltage in both magnitude and phase. Hence the characteristics of the line must be expressed in terms of complex quantities rather than the real quantities *r* and *g* used before. Resistance *r* must be replaced by the impedance $r + jx = Z\underline{/\psi}$ ohms per mile, and conductance *g* must be replaced by the admittance $g + jb = Y\underline{/\phi}$ mhos per mile. The equation of Example 3, Section 53, would then become $Z_o\underline{/\theta} = \sqrt{\frac{r + jx}{g + jb}} \tanh\left[\sqrt{(r + jx)\,(g + jb)}\,L\right]$. Here we have the hyperbolic tangent of the complex number in brackets. And the equation in Exercise 3, Section 53, would become

$$I = E\sqrt{\frac{g + jb}{r + jx}} \Big/ \sinh\left[\sqrt{(r + jx)\,(g + jb)}\,L\right].$$

Here we need to be able to evaluate the hyperbolic sine of the complex number in brackets.

So hyperbolic functions of complex numbers appear in electrical engineering work dealing with long lines having alternating voltages impressed. They also appear in studies of sound and acoustics and in studies of vibrations in long beams and other continuous media.

Review of calculations involving complex numbers.

Let us review a few concepts regarding complex numbers and some methods of calculating them.

1. A complex number w may be represented in component form or polar form.

   w = a + jb (Component form)
   w = $W\underline{/\alpha}$ (Polar form)

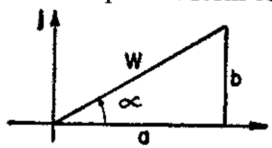


Fig. 65

   The correspondence between the two forms is shown graphically in Figure 65.

2. Complex numbers are added or subtracted by adding or subtracting their components. If w = a + jb and z = c + jd, then $w \pm z = (a + jb) \pm (c + jd) = (a \pm c) + j\,(b \pm d)$.
3. Complex numbers are more readily multiplied and divided when expressed in polar form. If
   $w = W\underline{/\alpha}$ and $z = Z\underline{/\theta}$, then
   $wz = (W\underline{/\alpha})\,(Z\underline{/\theta}) = (WZ)\,\underline{/\alpha + \theta}$, and
   $\frac{w}{z} = \frac{W\underline{/\alpha}}{Z\underline{/\theta}} = \left(\frac{W}{Z}\right)\underline{/\alpha - \theta}.$
4. Powers and roots of complex numbers are readily found when the numbers are expressed in polar form.
   If $w = W\underline{/\alpha}$, then $w^n = W^n\underline{/n\alpha}$ and $\sqrt[n]{w} = \sqrt[n]{W}\,\underline{/\alpha/n}$.

The two most frequent slide rule calculations arising from complex numbers are: (1) conversion from polar to component form, and (2) conversion from component to polar form.

1. Conversion from polar to component form.

Given $w = W\underline{/\alpha}$, find the component form w = a + jb. Reference to Figure 65 indicates that a = W cos ∝ and b = W sin ∝. The general method for the conversion from polar to component form then becomes:

a. Set index of slide to W on "D".
b. Opposite 90° − ∝ on "ST" or ∝ *red* on "S" read a on "D".
c. Opposite ∝ *black* on "ST" or "S" read b on "D".

Example 1. Find the component form of $w = 163\underline{/27.2^\circ}$.
Set right index of slide to 163 on "D".
Opposite 27.2° red on "S" read a = 145 on "D".
Set left index of slide to 163 on "D".
Opposite 27.2° black on "S" read b = 74.5 on "D".
$w = 163\underline{/27.2^\circ} =$ **145 + *j*74.5.**

Example 2. Find the component form of $w = 57.3\underline{/-4.28^\circ}$.
Set right index of slide to 57.3 on "D".
Opposite 4.28° red on "S" read a = 57.2 on "D".
Opposite 4.28° black on "ST" read b = 4.28 on "D".
$w = 57.3\underline{/-4.28^\circ} =$ **57.2 − j4.28.**

2. Conversion from component to polar form.

Given w = a + jb, find the polar form $w = W\underline{/\alpha}$. Reference to Figure 65 indicates that $\tan \alpha = \frac{b}{a}$, or that $\tan (90^\circ - \alpha) = \frac{a}{b}$, and that $W = \frac{b}{\sin \alpha}$. The general method for the conversion from component to polar form then becomes:

a. Find ∝.

   (1) If b is less than a, then tan ∝ < 1 and ∝ < 45°.
   Set index of slide to a on "D".
   Opposite b on "D" read ∝ on "T<45" (or "ST") black.

   (2) If b is greater than a, then tan ∝ > 1 and ∝ > 45°.
   Set index of slide to a on "D".
   Opposite b on "D" read ∝ on "T>45" red.

In general, set the index of the slide to a on "D". Opposite b on "D" read ∝ on "T<45", "T>45" or "ST". If b is less than a, ∝ is less than 45° and is read on the "T<45" scale. If b is greater than a, ∝ is greater than 45° and is read on the "T>45" scale.

b. With ∝ determined W is found from $W = \frac{b}{\sin \alpha}$. Bring ∝ on "S" (or "ST") to b on "D". Opposite index of slide read W on "D". (The relation $W = \frac{a}{\cos \alpha}$ may also be used to find W.)

Example 3. Find the polar form of $w = 12 + j9$.

Since $b < a$, therefore $\alpha < 45^\circ$.
Set left index of slide to 12 on "D".
Opposite 9 on "D" read $\alpha = 36.87^\circ$ on "T<45" black.
Bring 36.87° on "S" to 9 on "D".
Opposite left index of "D" read W = 15.0.
$w = 12 + j9 = \mathbf{15.0\,\angle 36.87^\circ}$.

Example 4. Find the polar form of $w = 2.11 - j3.78$.

Since $b > a$, therefore $\alpha > 45^\circ$.
Set left index of slide to 2.11 on "D".
Opposite 3.78 on "D" read $\alpha = 60.83^\circ$ on "T>45".
Bring 60.83° on "S" to 3.78 on "D".
Opposite right index of slide read W = 4.33.
$w = 2.11 - j3.78 = \mathbf{4.33\,\angle -60.83^\circ}$.

Example 5. Find the polar form of $w = 1.69 + j52.4$.

Since $b > a$, therefore $\alpha > 45^\circ$.
Set right index of slide to 52.4 on "D".
Opposite 1.69 on "D" read $90^\circ - \alpha = 1.85^\circ$ on "ST".
Hence, $\alpha = 90^\circ - 1.85^\circ = 88.15^\circ$ and $\sin\alpha = 1.00$ very nearly. Thus $W = \dfrac{b}{\sin\alpha} = b$, very nearly.
$w = 1.69 + j52.4 = \mathbf{52.4\,\angle 88.15^\circ}$.

For angles less than 0.573°, one may use the approximation $\tan\alpha = \alpha$, if α is converted to radians.

Calculations involving complex numbers are carried out by converting the complex numbers to the convenient form (component or polar) and by carrying out the indicated operations. As always, headwork will save time and reduce wear on the slide rule. And calculations can be checked by performing the operations in reverse sequence.

Example 6. An impedance $Z_1\angle\theta_1 = 12\angle 35.2^\circ$ ohms is added in series with an impedance $Z_2\angle\theta_2 = 9\angle -50.25^\circ$ ohms. Find the resultant impedance $Z\angle\theta$ in polar form.

Solution:

To add the impedances we convert to component form and add the components. $Z_1\angle\theta_1 + Z_2\angle\theta_2 = (R_1 + jX_1) + (R_2 + jX_2) = (R_1 + R_2) + j(X_1 + X_2)$.

Set left index of slide to 12 on "D".
Opposite 35.2° on "S" red read $R_1 = 9.80$ on "D".
Opposite 35.2° on "S" black read $X_1 = 6.92$ on "D".
$Z_1\angle\theta_1 = 12\angle 35.2^\circ = \mathbf{9.80 + j6.92}$ **ohms.**

Set right index to 9 on "D".
Opposite 50.25° on "S" red read $R_2 = 5.75$ on "D".
Opposite 50.25° on "S" black read $X_2 = 6.92$ on "D".
$Z_2\angle\theta_2 = 9\angle -50.25^\circ = 5.75 - j6.92$.
$Z\angle\theta = (9.80 + j6.92) - (5.75 - j6.92) = 15.55 + j0$ *ohms, or* $Z\angle\theta = \mathbf{15.55\,\angle 0^\circ}$ **ohms.**

Example 7. A telephone line has series impedance $z = 4.14 + j16.85$ ohms per mile and shunt admittance $y = 0.80 \times 10^{-6} + j45.7 \times 10^{-6}$ mhos per mile. Find the characteristic impedance $Z_0\angle\theta = \sqrt{\dfrac{z}{y}}$.

Solution:

To find Z_0 easily we need to convert z and y into polar form and then take

$$Z_0\angle\theta = \sqrt{\frac{Z\angle\psi}{Y\angle\phi}} = \sqrt{\frac{Z}{Y}}\angle \tfrac{1}{2}(\psi - \phi).$$

Find $z = Z\angle\psi$.

Set left index of slide to 16.85 on "D".
Opposite 4.14 on "D" read $\psi = 76.2^\circ$ on "T>45" black.
Bring 76.2° on "S" to 16.85 on "D".
Opposite right index of slide read Z = 17.36 on "D".
$z = 4.14 + j16.85 = 17.36\angle 76.2^\circ$ *ohms.*

Find $y = Y\angle\phi$.

Set left index of slide to 45.7×10^{-6} on "D".
Opposite 0.80×10^{-6} on "D" read $90^\circ - \phi = 1.00^\circ$ on "ST".

Then $Y = \dfrac{b}{\sin\phi} = 45.7 \times 10^{-6}$.

$Y = 0.80 \times 10^{-6} + j45.7 \times 10^{-6} = 45.7 \times 10^{-6}\angle 89.0^\circ$ *mhos.*

$$Z_0\angle\theta = \sqrt{\frac{Z}{Y}}\angle \tfrac{1}{2}(\psi - \phi).$$

Set 45.7×10^{-6} on B-RIGHT to 17.36 on A-RIGHT.
Opposite right index of slide read $Z_0 = 616$ ohms on "D".
$Z_0\angle\theta = 616\angle \tfrac{1}{2}(76.2^\circ - 89.0^\circ) = \mathbf{616\angle -6.4^\circ}$ **ohms.**

Exercises

1. Find the following complex numbers in component form.
   a. $29.7\angle 41.3^\circ$ c. $2\angle 2.54^\circ$
   b. $0.0261\angle -12.8^\circ$ d. $13.6\angle 123^\circ$

2. Find the polar form for the following complex numbers.
   a. $4 - j3$ c. $0.53 + j17.2$
   b. $0.0298 + j0.0617$ d. $-3.51 + j2.36$

3. The circuit in Figure 63 has $R = 4.2$ ohms and $L = 0.80$ henries. A 110-volt, 60-cycle voltage source E impressed on this circuit will cause a current $I\angle\phi = \dfrac{E\angle\psi}{Z\angle\theta}$ to flow, where $E = 110\angle 0^\circ$, and $Z\angle\theta = R + jX$. The reactance X is given by $wL = 377(0.8) = 301.6$ ohms. Find I in component form.

4. A telephone line has series impedance $z = 4.14 + j16.85$ ohms per mile and shunt admittance $y = 0.80 \times 10^{-6} + j45.7 \times 10^{-6}$ mhos per mile. Find the propagation constant $P\angle\alpha = \sqrt{zy}$ in component form.

Answers to the above exercises.

1. a. $22.3 + j19.6$ c. $1.998 + j0.0886$
   b. $0.0255 - j0.0058$ d. $-7.41 + j11.4$
2. a. $5\angle -36.9^\circ$ c. $17.2\angle 88.24^\circ$
   b. $0.0685\angle 64.2^\circ$ d. $4.23\angle 146.1^\circ$
3. $I\angle\phi = 0.0051 - j0.365$ amperes.
4. $P\angle\alpha = 0.00363 + j\,0.0280$.

55. Hyperbolic Functions of Complex Numbers.

As we have seen in the preceding review of complex numbers, the sum or difference of two complex numbers is another complex number. The product or quotient of two complex numbers leads to another complex number. Similarly, a power or root of a complex number is just a complex number. So it should not surprise us to discover that a hyperbolic function of a complex number is simply another complex number.

From the algebraic definitions of hyperbolic functions and from the expansions for the functions of the sum of two numbers, the following identities may be proved:

$$\sinh(u \pm j\theta) = \sinh u \cos\theta \pm j\cosh u \sin\theta = A\angle\pm\alpha$$
$$\cosh(u \pm j\theta) = \cosh u \cos\theta \pm j\sinh u \sin\theta = B\angle\pm\beta$$
$$\tanh(u \pm j\theta) = \frac{\tanh u \pm j\tan\theta}{1 \pm j\tanh u \tan\theta} = \frac{A\angle\pm\alpha}{B\angle\pm\beta} = \frac{A}{B}\angle\pm(\alpha - \beta)$$

To find the hyperbolic functions of a known complex number $u \pm j\theta$, one could calculate the components by multiplying the functions $\sinh u \cos\theta$, $\cosh u \sin\theta$, $\cosh u \cos\theta$ and $\sinh u \sin\theta$, but quicker methods for finding A and α, or B and β, more directly would be desirable. These we shall develop.

It may be noted in passing that the ordinary trigonometric functions of a complex number are themselves complex numbers.

$$\sin(\theta + ju) = \sin\theta\cosh u + j\cos\theta\sinh u$$
$$\cos(\theta + ju) = \cos\theta\cosh u - j\sin\theta\sinh u$$
$$\tan(\theta + ju) = \frac{\tan\theta + j\tanh u}{1 - j\tan\theta\tanh u}$$

A. Evaluation of sinh (u ± j θ).

From the expansion of $\sinh(u + j\theta)$ we obtain the graphical picture shown in Figure 66.

$$\sinh(u + j\theta) = \sinh u \cos\theta + j\cosh u \sin\theta = A\angle\alpha$$

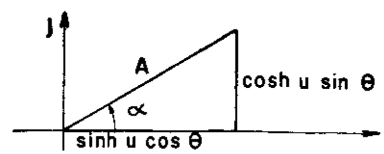


Fig. 66

From this we find:

$$\tan\alpha = \frac{\cosh u \sin\theta}{\sinh u \cos\theta} = \frac{\tan\theta}{\tanh u} = \tan\theta\coth u$$
$$A = \frac{\sinh u \cos\theta}{\cos\alpha}$$

The methods for finding α and *A* should be clear from these formulae.

But a few REMINDERS may facilitate the numerical calculations.

Let us evaluate sinh (1.3 + j0.57).

REMINDER 1. *The angle* $\theta = 0.57$ *is in radians and must be converted to degrees before it can be found on the "T<45" and "T>45" scales.* (See section 36.)

$$\frac{\theta}{0.57} = \frac{180^\circ}{\pi}$$

Set 180 on "CF" to π *right on DF.*
Opposite 0.57 on "DF" (or "D") read $\theta = 32.6^\circ$ *on "CF" (or "C").*

REMINDER 2. Since tanh u is always less than *1*, therefore the value of the ratio $\frac{\tan\theta}{\tanh u}$ will always be greater than $\tan\theta$. *Hence* $\tan\alpha$ *is always greater than* $\tan\theta$, *or* α *is always greater than* θ.

Set 32.6 on "T<45" to 1.3 on "Th".
Opposite right index of body read $\alpha = 36.6^\circ$ *on "T<45".*

As we predicted, α is greater than θ. The logic underlying this setting of the rule is shown in Figure 67, where distances have been labeled in terms of the quantities they represent. A distance corresponding to coth 1.3 $\left(\text{or } \frac{1}{\tanh 1.3}\right)$ has been added to the tan 32.6° distance along the "T<45" scale to give the tan α distance along the "T<45" scale. That the indicated distance does actually correspond to coth 1.3 can be verified by referring back to Section 52 where the method for determining coth u was explained.

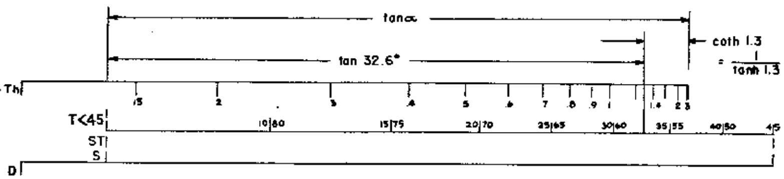


Fig. 67

REMINDER 3. Since α is always greater than θ, therefore $\cos\alpha$ is always less than $\cos\theta$. *Hence* $A = \sinh u \left(\frac{\cos\theta}{\cos\alpha}\right)$ *is always greater than sinh u.*

Continuing the above example, we have

$$A = \sinh 1.3 \left(\frac{\cos 32.6^\circ}{\cos 36.6^\circ}\right)$$

Set red 36.6° on "S" to 1.3 on "Sh2".
Opposite red 32.6° on "S" read A = 1.783 on "D".
Thus $\sinh(1.3 + j0.57) = 1.783\underline{/36.6^\circ}$, *or*
$\sinh(1.3 + j0.57) = 1.43 + j1.06$ *in component form.*

Example 1. Evaluate sinh (0.295 + j0.026).

Set 180 on "CF" on π *right on "DF".*
Opposite 0.026 on "D" read $\theta = 1.49^\circ$ *on "C".*
Set 1.49° on "ST" to 0.295 on "Th".
Opposite right index of body read $\alpha = 5.20^\circ$ *on "ST".*

Set red 5.20° on "S" to 0.295 on "Sh1".
Opposite red 1.49° on "S" read A = 0.300 on "D".
Sinh (0.295 + j0.026) = **$0.300\underline{/5.20^\circ}$**.

In determinations of α, it may happen that the addition of the coth u distance to the $\tan\theta$ distance brings the right index of the body beyond the right end of the tangent scale on the slide. For instance, in the evaluation of α for sinh (0.187 + j3.61°) the setting of the rule is illustrated in Figure 68.

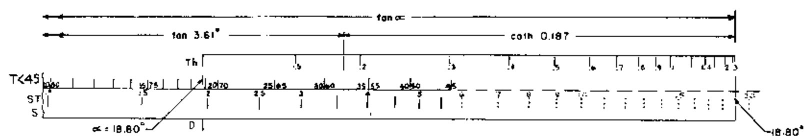


Fig. 68

We should like to read the answer on an extension of the "ST" scale to larger angles. The rightward extension of the "ST" scale, shown dotted in Figure 68, is a fictitious scale laid out just like the "T<45" scale. As a matter of fact, the "T<45" scale would be this extension of the "ST" scale, but it was cut off and slid back one scale length to lie along the "C" scale. Because of this fact the angle which would have been read on the dotted "T<45" scale opposite the right index of the body in Figure 68 can now be found on the true "T<45" scale opposite the left index of the body. Setting the hairline to the left index on the body, we read under the hairline on the "T<45" scale, α = *18.80°*. Continuing the calculation as before, we find that sinh (0.187 + j3.61°) = $0.198\underline{/18.80^\circ}$.

Example 2. Evaluate sinh (0.831 + j0.0911).

Set 180 on "CF" to π right on "DF".
Opposite 0.0911 on "DF" read $\theta = 5.22°$ on "CF".
Set 5.22 on "ST" to 0.831 on "Th".
Opposite left index of body read $\propto = 7.61°$ on "T<45".
Set red 7.61° on "S" to 0.831 on "D".
Opposite red 5.22° on "S" read A = 0.934 on "D".
Sinh (0.831 + j0.0911) = **0.934/7.61°**.

In some instances it may also happen that the addition of the coth u distance to the tan θ distance will bring the right body index out past the end of the "T<45" scale. Because of the double "T" scale, the answer can be read directly on the "T>45" scale, since the "T>45" scale is simply an extension of the "T<45" scale that has been cut off and slid back one scale length to lie along the "C" scale.

Example 3. Evaluate sinh (0.251 + j0.733).

Set 180 on "CF" to π right on "DF".
Opposite 0.733 on "DF" read $\theta = 42.0°$ on "CF".
Set black 42.0° on "T<45" to 0.251 on "Th".
Opposite left hand "D" index read $\propto = 74.7°$ on "T>45".
Set red 74.7° on "S" to 0.251 on "Sh1".
Opposite red 42.0° on "S" read A = 0.716 on "D".
Sinh (0.251 + j0.733) = **0.716/74.7°**.

We have not considered a case in which θ is greater than 45°, but the principles involved are exactly the same. To find $\tan \propto = \tan\theta \coth u$, we add the coth u distance to the tan θ distance and obtain the tan $\propto$ distance.

Example 4. Evaluate sinh (0.836 + j0.904).

Set 180 on "CF" to π right on "DF".
Opposite 0.904 on "DF" read $\theta = 51.75°$ on "CF".
Set red 51.75° on "T>45" to 0.836 on "Th".
Opposite right hand "D" index read $\propto = 61.65°$ red on "T>45".
Set red 61.65° on "S" to 0.836 on "Sh1".
Interchange indices of the slide.
Opposite red 51.75° on "S" read A = 1.22 on "D".
Sinh (0.836 + j0.904) = **1.22/61.65°**.

When θ is greater than 45°, it may occur that $\propto$ is greater than 84.4°, so that it cannot be found on the rule. In such cases we might interchange indices of the slide and work back along the "ST" scale, taking complementary angles. But a safer procedure, until one has mastered the rule well enough to work out his own settings, is to use the fundamental definition as suggested below for special cases.

In special cases where the values of u or θ or $\propto$ do not appear on the slide rule, we can always resort to the fundamental definition:

$$\sinh(u + j\theta) = \sinh u \cos\theta + j \cosh u \sin\theta.$$

Since we can always evaluate sinh u, cosh u, cos θ, and sin θ, the value of sinh $(u + j\theta)$ can always be found in component form in accordance with the above definition. Then the complex number may be converted to polar form in the usual manner.

A few special cases will be mentioned and the methods for handling them indicated.

1. For $\sinh(u - j\theta)$ we obtain by expansion

$$\begin{aligned}\sinh(u - j\theta) &= \sinh u \cos(-\theta) + j \cosh u \sin(-\theta)\\ &= \sinh u \cosh\theta - j \cosh u \sin\theta = A\underline{/-\propto}.\end{aligned}$$

Hence, the evaluation may be carried out on the slide rule as before, but the angle $\propto$ will be negative. For instance, if Example 3 had been the evaluation of sinh(0.251 − j0.733), the answer would have been 0.716/−74.7°.

2. For $\sinh(-u + j\theta)$ we obtain by expansion

$$\begin{aligned}\sinh(-u + j\theta) &= \sinh(-u)\cos\theta + j\cosh(-u)\sin\theta\\ &= -\sinh u \cos\theta + j \cosh u \sin\theta\\ &= -(\sinh u \cos\theta - j \cosh u \sin\theta)\end{aligned}$$

Hence $\sinh(-u + j\theta) = -\sinh(u - j\theta) = -A\underline{/-\propto}$. The evaluation of the latter expression was explained in (1) above.

3. For angles θ greater than 90° we may use trigonometric reduction formulae and change the problem to one we have already solved. Write θ as $180° \pm \phi$, where ϕ is less than 90°. Then

$$\begin{aligned}\sinh(u + j\theta) &= \sinh u \cos\theta + j \cosh u \sin\theta\\ &= \sinh u \cos(180° \pm \phi) + j \cosh u \sin(180° \pm \phi)\\ &= -\sinh u \cos\phi \mp j \cosh u \sin\phi\\ &= -(\sinh u \cos\phi \pm j \cosh u \sin\phi)\\ &= -A\underline{/\pm\propto}.\end{aligned}$$

Hence $\sinh(u + j\theta) = -\sinh(u \pm j\phi) = -A\underline{/\pm\propto}$, and we know how to evaluate the expression from (1) and (2) above.

4. For θ less than 0.6° we may take $\sin\ \theta = \tan\ \theta = \theta$ (in radians) and $\cos\ \theta = 1$. The values of these functions may then be set directly on the "C" and "D" scales without resort to the "S", "T<45", "T>45", and "ST" scales. Similarly, if u is less than 0.1, we may take $\sinh u = \tanh u = u$ and can set these values directly on the "C" and "D" scales.

5. For θ between 84.3° and 90° we may take $\sin\ \theta = 1.00$, $\cos\ \theta = \sin(90° - \theta)$, and $\tan\ \theta = \cot(90° - \theta) = \dfrac{1}{\tan(90° - \theta)}$. Then we may evaluate $\sin(90° - \theta)$ and $\tan(90° - \theta)$ from the "ST" scale or by using the approximations in (4) above. Similarly, if u is greater than 3, so that sinh u cannot be read on the rule, we may find the value of sinh u as explained in Section 53 and set this value directly on the "D" scale.

Example 5. Evaluate $\sinh(4.60 - j1.37)$.

Set 180 on "CF" to π right on "DF".
Opposite 1.37 on "DF" read $\theta = 78.5°$ on "CF".
Since tanh 4.60 = 1, very closely, $\tan \alpha = \tan\ \theta$, or $\alpha = \theta = 78.5°$.
Find $\sinh 4.60 = \frac{1}{2}e^{4.60}$.
Set hairline to 4.60 on "D".
Under hairline on "LL3" read $e^{4.60} = 100$.
Divide by 2 in your head, and obtain sinh 4.60 = 50.
$$A = \sinh u \frac{\cos\ \theta}{\cos \alpha} = 50, \text{ since } \theta = \alpha.$$
Hence, $\sinh(4.60 - j1.37) =$ $\mathbf{50\angle{-78.5°}}$.

Example 6. Evaluate $\sinh(1.21 + j2.86)$.

Set 180 on "CF" to π right on "DF".
Opposite 2.86 on "DF" read $\theta = 163.8°$ on "CF".
Let $\theta = 180° - \phi$, then $\phi = 180° - 163.8° = 16.2°$.
From (3) above, we have $\sinh(1.21 + j2.86) = -\sinh(1.21 - j16.2°)$ and we proceed to solve as usual.
Set 16.2° on "T<45" to 1.21 on "Th".
Opposite right body index read $\alpha = 19.13°$ on "T<45".
Set red 19.13° on "S" to 1.21 on "Sh2".
Opposite red 16.2° on "S" read A = 1.55 on "D".
Hence, $\sinh(1.21 + j2.86) =$ $\mathbf{-1.55\angle{-19.13°}}$.

Example 7. Evaluate $\sinh(0.0024 + j0.00195)$.

$\sinh(0.0024 + j0.00195)$
$= \sinh 0.0024 \cos 0.00195 + j \cosh 0.0024 \sin 0.00195$
$= (0.0024)\ (1.00) + j(1.00)\ (0.00195)$
$= \mathbf{0.0024 + j0.00195}$.

For Example 7 we see that for u and θ less than 0.100 the value of $\sinh(u + j\theta) = u + j\theta$ very nearly.

Exercises

1. Evaluate the following:
   - a. $\sinh(0.388 + j0.317)$
   - b. $\sinh(0.0162 - j0.0275)$
   - c. $\sinh(1.11 + j1.36)$
   - d. $\sinh(0.231 - j0.700)$
   - e. $\sinh(1.47 + j1.47)$
   - f. $\sinh(2.02 - j1.92)$

2. Check the first four problems above by calculating the value of $\sinh(u + j\theta)$ in component form from the definition $\sinh(u + j\theta) = \sinh u \cos\ \theta + j \cosh u \sin\ \theta$.

3. A telephone line has series impedance $z = 4.14 + j16.85$ ohms per mile and shunt admittance $y = 0.80 \times 10^{-6} + j45.7 \times 10^{-6}$ mhos per mile. The current I in amperes flowing through a short-circuit 67 miles from the source end of the line is given by
$$I = E\sqrt{\frac{y}{z}}\frac{1}{\sinh\sqrt{zy}\,x},$$
where $E = 24\angle{0°}$ volts and $x = 67$ miles. Find I.

Answers to the above exercises.

1. a. $0.505\angle{41.6°}$ b. $0.032\angle{-59.5°}$ c. $1.67\angle{80.22°}$ d. $0.684\angle{-74.92°}$ e. $2.28\angle{84.76°}$ f. $-3.82\angle{-70.65°}$
2. a. $0.378 + j0.336$ b. $0.162 - j0.0275$ c. $0.284 + j1.64$ d. $0.178 - j0.660$
3. $I = -0.0392\angle{92.1°}$ amperes.

B. *Evaluation of cosh $(u + j\theta)$.*

From the expansion of $\cosh(u + j\theta)$ we find that it is simply a complex number which can be indicated graphically as shown in Figure 70.

$$\cosh(u + j\theta) = \cosh u \cos\ \theta + j \sinh u \sin\ \theta = B\angle{\beta}$$

From this picture we obtain:

$$\tan \beta = \frac{\sinh u \sin \theta}{\cosh u \cos \theta} = \frac{\tan \theta}{\coth u}$$

$$B = \frac{\sinh u \sin \theta}{\sin \beta}$$

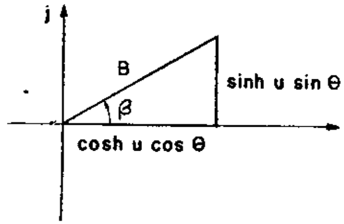


Fig. 70

The methods for finding β and B from these formulae are quite similar to the methods used for evaluating $\sinh(u + j\theta)$ in the preceding pages. If a few REMINDERS are kept firmly in mind, the calculations can be made quickly and without error.

REMINDER 1. *The angle θ must be converted to degrees before it is located on the slide rule.*

REMINDER 2. Since coth u is always greater than 1, therefore tan β is always less than tan θ. *Hence β is always less than θ.*

REMINDER 3. Since β is less than θ, therefore the ratio $\frac{\sin \theta}{\sin \beta}$ is always greater than 1. *Hence B is always greater than sinh u.* In determinations of B the decimal point may be correctly placed by noting the value of sinh u and by multiplying it by an approximate value of $\frac{\sin \theta}{\sin \beta}$.

As an example let us evaluate $\cosh(0.482 + j0.657)$. First, we must convert θ to degrees.

Set 180 on "CF" to π right on "DF".
Opposite 0.657 on "DF" read $\theta = 37.6°$ on "CF".

Next, to find $\tan \beta = \frac{\tan \theta}{\coth u}$ we wish to subtract the coth u distance on the rule from the tan θ distance to obtain the tan β distance.

Set 37.6° on "T<45" to right index of body.
Opposite 0.482 on "Th" read $\beta = 19.1°$ on "T<45".

The setting of the rule for this determination of β is shown in Figure 71.

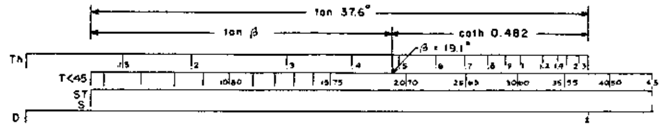


Fig. 71

Finally, to determine $B = \sinh u \left(\frac{\sin \theta}{\sin \beta}\right)$ the following settings are used.

Set 19.1° on black "S" scale to 0.482 on "Sh1".
Opposite 37.6° on black "S" scale read B = 0.940 on "D".

The decimal point is placed by noting that sinh 0.482 is about 0.5 and $\frac{\sin 37.6°}{\sin 19.1°}$ is about 2. Hence $\cosh(0.482 + j0.657) = 0.940\underline{/19.1°}$.

If θ is found on the "ST" scale, the angle β may be found by a setting similar to that given above.

Example 8. Evaluate $\cosh(0.831 + j5.22°)$.

Set 5.22° on "ST" to right body index.
Opposite 0.831 on "Th" read $\beta = 3.56°$ on "ST".
Set 3.56° on "ST" to 0.831 on "Sh1".
Interchange indices of the rule.
Opposite 5.22° on "ST" read B = 1.36 on "D".
Cosh (0.831 + j5.22°) = **1.36$\underline{/3.56°}$**.

In the determination of β from the relation $\tan \beta = \frac{\tan \theta}{\coth u}$, the subtraction of the coth u distance from the tan θ distance may locate an answer off the left end of the slide. Such a setting is shown below for the determination of β in the evaluation of $\cosh(0.126 + j27.5°)$.

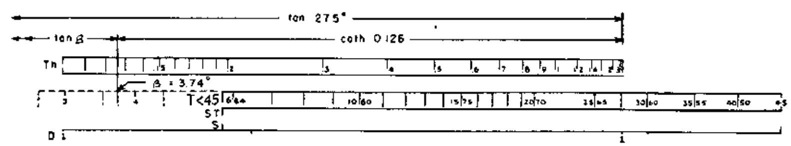


Fig. 72

Opposite 0.126 on the "Th" scale we should like to find the value of β on a leftward extension of the "T<45" scale to smaller angles. This fictitious extension of the "T<45" scale, shown dotted in Figure 72, is similar to the "ST" scale. So we can simply interchange indices of the slide and bring the "ST" scale into the position of the dotted scale of Figure 72. Having done so, we read $\beta = 3.74°$ on the "ST" scale opposite 0.126 on "Th". Continuing the calculation in the usual manner, we find $\cosh(0.126 + j27.5°) = 0.895\underline{/3.74°}$.

If the angle θ is on the "ST" scale originally and subtraction of the coth u distance leads to an answer off the left end of that scale, we interchange indices as before but read the angle as one-tenth of the value found on the "ST" scale opposite the value of u on "Sh".

Example 9. Evaluate $\cosh(0.026 + j0.037)$.

Set 180 on "CF" to π right on "DF".
Opposite 0.037 on "D" read $\theta = 2.12°$ on "C".
Since $u < 0.1$, take $\tanh u = \sinh u = u$.
Set 2.12° on "ST" to right index of body.
Interchange indices of the slide.
Opposite 0.026 on "D" read $\beta = 0.0551°$ on "ST".
Set 0.0551° on "ST" to 0.026 on "D".
Opposite 2.12° on "ST" read $B = 1.00$ on "D".
$\cosh(0.026 + j0.037) = $ $\mathbf{1.00\underline{/0.0551°}}$.

(The decimal point can be located properly in B by noting that $B = \sinh u \left(\frac{\sin\theta}{\sin\beta}\right)$ is given approximately by $u\left(\frac{\theta}{\beta}\right) = 0.026\left(\frac{2.12}{0.0551}\right) = 1.00$.)

A third type of problem which may arise in evaluation of cosh $(u + j\theta)$ is where $\theta > 45$. The basic formula for finding β is the same as before: $\tan\beta = \dfrac{\tan\theta}{\coth u}$

Example 10. Evaluate $\cosh(0.174 + j1.37)$.

Set 180 on "CF" to π right on "DF".
Opposite 1.37 on "DF" read $\theta = 78.5$ on "CF".
Set left index to .174 on "Th".
Move hairline to 78.5 on "T>45".
Close the rule and read $\beta = 40.25°$ under the hairline on "T<45".

In special cases where values of u or θ or β do not appear on the slide rule, we can always resort to the fundamental definition:

$$\cosh(u + j\theta) = \cosh u \cos\theta + j \sinh u \sin\theta$$

Since we can always evaluate cosh u, sinh u, cos θ, and sin θ, the value of $\cosh(u + j\theta)$ can always be found in component form in accordance with the above definition. Then the complex number may be converted into polar form in the usual manner.

Some special cases can readily be handled without resort to the fundamental definition.

1. For $\cosh(u - j\theta)$ we obtain by expansion
$$\begin{aligned}\cosh(u - j\theta) &= \cosh u \cos(-\theta) + j \sinh u \sin(-\theta)\\ &= \cosh u \cos\theta - j \sinh u \sin\theta = B\underline{/-\beta}.\end{aligned}$$
Hence, the evaluation may be carried out on the slide rule as before, but the angle β will be negative. For instance, if Example 10 had been the evaluation of $\cosh(0.174 - j78.5°)$, the answer would have been $0.265\underline{/-40.25°}$.

2. For $\cosh(-u + j\theta)$ we obtain by expansion
$$\begin{aligned}\cosh(-u + j\theta) &= \cosh(-u)\cos\theta + j\sinh(-u)\sin\theta\\ &= \cosh u \cos\theta - j \sinh u \sin\theta = B\underline{/-\beta}.\end{aligned}$$
The evaluation can be carried out on the slide rule in the normal manner and the value of β made negative. If Example 10 had been the evaluation of $\cosh(-0.174 + j78.5°)$, the answer would have been $0.265\underline{/-40.25°}$.

3. For angles θ greater than 90° we may use trigonometric reduction formulae and change the problem to one we have already solved. Write θ as $180° \pm \phi$, where ϕ is less than 90°. Then
$$\begin{aligned}\cosh(u + j\theta) &= \cosh u \cos\theta + j \sinh u \sin\theta\\ &= \cosh u \cos(180° \pm \phi) + j \sinh u \sin(180° \pm \phi)\\ &= -\cosh u \cos\phi \mp j \sinh u \sin\phi\\ &= -(\cosh u \cos\phi \pm j \sinh u \sin\phi)\\ &= -\cosh(u \pm j\phi) = -B\underline{/\pm\beta}.\end{aligned}$$
Hence, having determined ϕ and its sign, we can evaluate B and β by the usual slide rule method and attach the correct signs to the result.

For special cases in which the values of u or θ are less or greater than values shown on the slide rule, the methods explained under special cases (4) and (5) in Section 55-A are applicable to evaluations of $\cosh(u + j\theta)$ also. Examples 5, 6, and 7 give applications of those methods.

Exercises

1. Evaluate the following:

   a. $\cosh(0.388 + j0.317)$
   b. $\cosh(0.276 + j0.213)$
   c. $\cosh(1.11 + j1.36)$
   d. $\cosh(0.729 - j0.916)$
   e. $\cosh(0.0162 - j0.0275)$
   f. $\cosh(4.23 + j3.68)$

2. Check the first four problems above by calculating the value of $\cosh(u + j\theta)$ in component form from the definition

$$\cosh(u + j\theta) = \cosh u \cos\theta + j \sinh u \sin\theta.$$

Answers to the above exercises.

1. a. $1.03\underline{/6.91°}$
   b. $1.016\underline{/3.33°}$
   c. $1.37\underline{/75.07°}$
   d. $1.00\underline{/-39.05°}$
   e. $1.00\underline{/-0.0255°}$.
   f. $-34.4\underline{/30.9°}$

2. a. $1.022 + j0.124 = 1.03\underline{/6.91°}$
   b. $1.015 + j0.0590 = 1.016\underline{/3.33°}$
   c. $0.353 + j1.323 = 1.37\underline{/75.07°}$
   d. $0.777 - j0.631 = 1.00\underline{/-39.05°}$

C. *Evaluation of tanh $(u + j\theta)$.*

The value of $\tanh(u + j\theta)$ is a complex number, which we shall designate in polar form as $C\underline{/\gamma}$. Since $\tanh(u + j\theta) = \dfrac{\sinh(u + j\theta)}{\cosh(u + j\theta)}$, we could evaluate it as $C\underline{/\gamma} = \dfrac{A\underline{/\alpha}}{B\underline{/\beta}} = \dfrac{A}{B}\underline{/\alpha - \beta}$, where A and α, B and β can be determined as explained in Section 55-A and 55-B. However, a shorter method for evaluating tanh $(u + j\theta)$ results from manipulation of its expanded form.

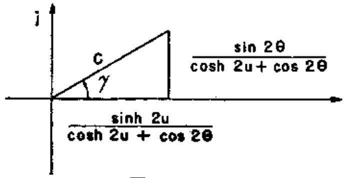


Fig. 73

$$\tanh(u + j\theta) = \frac{\sinh 2u + j \sin 2\theta}{\cosh 2u + \cos 2\theta} = C\underline{/\gamma}.$$

From this graphical representation of the hyperbolic tangent we obtain:

$$\tan\gamma = \frac{\sin 2\theta}{\sinh 2u}$$

$$C = \frac{\sqrt{\sinh^2 2u + \sin^2 2\theta}}{\cosh 2u + \cos 2\theta} = \sqrt{\frac{\cosh 2u - \cos 2\theta}{\cosh 2u + \cos 2\theta}}$$

The expression for C results from trigonometric manipulation after use of the Pythagorean theorem for the above triangle. The methods for finding C and γ from these formulae are perfectly straightforward, but a few REMINDERS may help to ensure speed and accuracy in the calculations.

REMINDER 1. *The angle θ must be converted to degrees before it is located on the slide rule.*

REMINDER 2. Since sinh 2u may be either greater than or less than 1, the value of γ may be less than or greater than 2θ. *The value of tan $\gamma = \dfrac{\sin 2\theta}{\sinh 2u}$ should be determined, and then γ should be determined by a new setting of the slide.*

REMINDER 3. *If the angle 2θ does not lie in the first quadrant, then care must be taken to give values of sin 2θ and cos 2θ their proper signs in calculating both γ and C.*

Consider the evaluation of $\tanh(0.836 + j0.904)$.

First we find 2θ and $2u$.

Set 180 on "CF" to π right on "DF".
Opposite 0.904 on "DF" read θ = 51.75° on "CF".
By mental calculation, 2θ = 103.5° and 2u = 1.672.

Next we find $\tan\gamma = \dfrac{\sin 2\theta}{\sinh 2u}$ and then obtain γ.

Sin 103.5° = sin(180° − 76.5°) = sin 76.5°.
Set 76.5° on black "S" scale to 1.672 on "Sh2".
Opposite left body index read tan γ = 0.379 on "C".
(The decimal point is located by noting that $\dfrac{\sin 76.5°}{\sinh 1.672}$ is approximately $\dfrac{1.00}{2.50} = 0.40$.)
Set hairline to 0.379 on "C".
Under hairline on "T<45" scale read γ = 20.75°.

Finally we calculate $C = \sqrt{\dfrac{\cosh 2u - \cos 2\theta}{\cosh 2u + \cos 2\theta}}$

Set right index of slide to 1.672 on "Th":
Opposite 1.672 on "Sh2" read cosh 1.672 = 2.76 on "C".
Cos 103.5° = cos(180° − 76.5°) = − cos 76.5°.
Set hairline to red 76.5° on "S".
Under hairline on "C" read cos 76.5° = 0.234.

$$C = \sqrt{\frac{2.76 - (-0.234)}{2.76 + (-0.234)}} = \sqrt{\frac{2.994}{2.526}}$$

Set 2.526 on B-LEFT to 2.994 on A-LEFT.
Opposite left index of slide read C = 1.089 on "D".
Tanh(0.836 + j0.904) = **1.089/20.75°**.

If the values of *u* and θ are small, say less than 0.2, then values of cosh 2u and cos 2 θ are both near 1.00 and the difference (cosh 2u − cos 2 θ) used in finding *C* can hardly be determined accurately. The fundamental equation should then be used.

$$\tanh(u + j\theta) = \frac{\sinh 2u + j \sin 2\theta}{\cosh 2u + \cos 2\theta} = C\underline{/\gamma}$$

A sample calculation based on this equation is given in Example 11 below.

To find the value of tanh (u − j θ), we can replace θ by − θ in the above formulae.

$$\tanh(u - j\theta) = \frac{\sinh 2u + j \sin(-2\theta)}{\cosh 2u + \cos(-2\theta)} = \frac{\sinh 2u - j \sin 2\theta}{\cosh 2u + \cos 2\theta}$$

$$\tanh(u - j\theta) = C\underline{/-\gamma}.$$

Special cases can be handled by methods explained in Sections 55-A and 55-B. One can always make use of the equation:

$$\tanh(u + j\theta) = \frac{\sinh 2u + j \sin 2\theta}{\cosh 2u + \cos 2\theta}.$$

Example 11. Evaluate tanh(0.836 + j0.904) by using the fundamental equation.

Set 180 on "CF" to π right on "DF".
Opposite 0.904 on "DF" read θ = 51.75° on "CF".
By mental calculation 2 θ = 103.5° and 2u = 1.672.
Hence tanh(0.836 + j0.904) =

$$\frac{\sinh 1.672 + j \sin 103.5^\circ}{\cosh 1.672 + \cos 103.5^\circ} = \frac{\sinh 1.672 + j \sin 76.5^\circ}{\cosh 1.672 - \cos 76.5^\circ}.$$

Set right index of slide to 1.672 on "Th".
Opposite 1.672 on "Sh2" read cosh 1.672 = 2.75 on "C" and sinh 1.672 = 2.57 on "D".
Set hairline to red 76.5° on "S" and read cos 76.5° = 0.223 on "C".
Cosh 1.672 + cos 103.5° = 2.75 − 0.223 = 2.527.
Set 2.527 on "C" to left body index.
Opposite 2.57 on "C" read 1.017 on "D".
Opposite sin76.5° on "S" read 0.385 on "D".
Hence tanh(0.836 + j0.904) = 1.017 + j0.385.
To convert to polar form set left index of slide to 1.017 on "D".
Opposite 0.385 on "D" read γ = 20.75° on "T<45".
Bring 20.75° on black "S" scale to 0.385 on "D".
Opposite left index of slide read C = 1.089 on "D".
Tanh(0.836 + j0.904) = **1.089/20.75°**.

Exercises

1. Evaluate the following.
   - a. tanh(0.388 + j0.317)
   - b. tanh(0.729 − j0.916)
   - c. tanh(0.231 + j0.700)
   - d. tanh(2.02 − j1.92)
2. Check the first two problems above by calculating the value of tanh(u + j θ) in component form from the relation $\tanh(u + j\theta) = \dfrac{\sinh 2u + j \sin 2\theta}{\cosh 2u + \cos 2\theta}$.
3. A telephone line has series impedance z = 4.14 + j16.85 ohms per mile and shunt admittance y = $0.80 \times 10^{-6} + j45.7 \times 10^{-6}$ mhos per mile. The impedance at one end of the line with the other end short-circuited is given by
$$Z_1 = \sqrt{\frac{z}{y}} \tanh \sqrt{zy}\, L,$$
where L is the length of the line in miles. Find Z_1, if L = 50 miles.

Answers to the above exercises.

1. a. $0.490\underline{/34.7^\circ}$ c. $0.856\underline{/64.1^\circ}$
   b. $1.12\underline{/-25.43^\circ}$ d. $1.03\underline{/-1.30^\circ}$
2. a. $0.403 + j0.279 = 0.490\underline{/34.7^\circ}$.
   b. $1.01 - j0.482 = 1.12\underline{/-25.43^\circ}$.
3. $Z_1 = 2{,}445\underline{/36.15^\circ}$ ohms.

56. Inverse Hyperbolic Functions of Complex Numbers.

Sometimes the value of a hyperbolic function of some complex number is known, and it is desired to determine the number. For instance, the terminal impedance at one end of an electric line when a short-circuit has occurred x miles from the source is given by

$$Z_1 = \sqrt{\frac{z}{y}}\tanh\sqrt{zy}\,x.$$

If the characteristic impedance z and admittance y are known for the line and if Z_1 is measured, then the distance out to the short-circuit can be determined.

$$\tanh\sqrt{zy}\,x = Z_1\sqrt{\frac{y}{z}}$$

$$x = \frac{1}{\sqrt{zy}}\tanh^{-1}\left(Z_1\sqrt{\frac{y}{z}}\right)$$

In this case, we need to find the complex number whose hyperbolic tangent is given as $Z_1\sqrt{\frac{y}{z}}$.

Let us develop methods for finding a complex number $u + j\theta$ when the value of its hyperbolic sine, cosine, or tangent is known. Evaluation of these three hyperbolic functions are sufficient, for the remaining three functions are the reciprocals of those mentioned above.

A. Finding u + j θ when sinh (u + j θ) is given.

Let $\sinh(u + j\theta) = A\underline{/\alpha} = a_1 + ja_2$. Trigonometric juggling leads to the following relation between the angle θ and the values a_1, a_2.

$$\sin\theta = \frac{\dfrac{2a_2}{a_1}}{\sqrt{1+\left(\dfrac{1+a_2}{a_1}\right)^2}+\sqrt{1+\left(\dfrac{1-a_2}{a_1}\right)^2}}$$

After θ has been determined from this equation the value of u can be determined from the relation

$$\sinh u = \frac{a_1}{\cos\theta}.$$

The methods of finding θ and u from these formulae are perfectly straightforward, but an example may bring out techniques for speeding up the calculations.

Example 1. Given $\sinh(u + j\theta) = 0.505\underline{/41.6^\circ}$,
find $u + j\theta = \sinh^{-1}(0.505\underline{/41.6^\circ})$.

Convert $\sinh(u + j\theta) = 0.505\underline{/41.6^\circ}$ *to component form.*
Set right index of slide to 0.505 on "D".
Opposite red 41.6° on "S" read $a_1 = 0.378$ *on "D".*
Opposite black 41.6° on "S" read $a_2 = 0.335$ *on "D".*
Hence $\sinh(u + j\theta) = a_1 + ja_2 = 0.378 + j0.335$.

$$\sin\theta = \frac{\dfrac{2(0.335)}{0.378}}{\sqrt{1+\left(\dfrac{1.335}{0.378}\right)^2}+\sqrt{1+\left(\dfrac{+0.665}{0.378}\right)^2}}$$

Set 0.378 on "C" to 1.335 on "D".
Opposite right index of slide read
$\left(\frac{1.335}{0.378}\right)^2 = 12.5$ *on "A".*
Add 1.00 mentally and set hairline to 13.5 on "A".
Under hairline on "D" read $\sqrt{1+\left(\frac{1.335}{0.378}\right)^2} = 3.675.$
Set 0.378 on "C" to 0.665 on "D".
Opposite left index of slide read
$\left(\frac{+0.665}{0.378}\right)^2 = 3.10$ *on "A".*
Add 1.00 mentally and set hairline to 4.10 on "A".
Under hairline on "D" read
$\sqrt{1+\left(\frac{+0.665}{0.378}\right)^2} = 2.024.$

$$\sin\theta = \frac{\dfrac{2(0.335)}{0.378}}{3.675 + 2.024} = \frac{0.670}{0.378(5.699)}.$$

Set 0.378 on "CI" to 5.699 on "D".
Opposite 0.670 on "D" read $\theta = 18.15^\circ$ *on black "S" scale.*
To convert θ *to radians set 180 on "CF" to* π *right on "DF".*
Opposite 18.15 on "C" read 0.317 on "D".

$$\sinh u = \frac{a_1}{\cos\theta} = \frac{0.378}{\cos 18.15^\circ}.$$

Set red 18.15° on "S" to 0.378 on "D".
Opposite right index of slide read $u = 0.388$ *on "Sh1".*
(The decimal point is determined for the value of u by a mental estimate of the value of sinh u, and hence of u.)
$u + j\theta = \sinh^{-1}(0.505\underline{/41.6^\circ}) = \mathbf{0.388 + j0.317}.$

If desired, the complex number $u + j\theta$ may be converted to polar form in the usual manner.

If α is negative, then θ will be negative. For example, $\sinh^{-1}(0.505\angle{-41.6°}) = 0.388 - j0.317$.

B. *Finding u + j θ when cosh (u + j θ) is given.*

Let $\cosh(u + j\theta) = B\angle\beta = b_1 + jb_2$. In forms convenient for slide rule calculations the formulae relating u and θ to b_1 and b_2 are as follows:

$$\cos\theta = \frac{\dfrac{2b_1}{b_2}}{\sqrt{1+\left(\dfrac{1+b_1}{b_2}\right)^2}+\sqrt{1+\left(\dfrac{1-b_1}{b_2}\right)^2}}$$

$$\sinh u = \frac{b_2}{\sin\theta}$$

Given the values of b_1 and b_2, the calculations based on these formulae are so similar to the calculations in Section 56-A that no example need be given.

If β is negative, then θ must be taken negative.

C. *Finding u + j θ when tanh (u + j θ) is given.*

Let $\tanh(u + j\theta) = C\angle\gamma = c_1 + jc_2$. Convenient formulae for determining u and θ are as follows.

$$\tan 2\theta = \frac{2c_2}{1 - C^2}$$

$$\tanh 2u = \frac{2c_1}{1 + C^2}$$

The calculations involved are straightforward, but one REMINDER is needed. *The values determined initially are 2θ and $2u$.* These values must be halved to obtain θ and u.

Example 2. Given $\tanh(u + j\theta) = 3.98\angle 42.6°$, find $u + j\theta = \tanh^{-1}(3.98\angle 42.6°)$.

Convert $\tanh(u + j\theta) = 3.98\angle 42.6°$ to component form.
Set right index of slide to 3.98 on "D".
Opposite red 42.6° on "S" read $c_1 = 2.93$ on "D".
Opposite black 42.6° on "S" read $c_2 = 2.70$.
Hence $\tanh(u + j\theta) = 3.98\angle 42.6° = 2.93 + j2.70$.
Set hairline to 3.98 on "D".

Under hairline on "A" read $C^2 = 15.85$.

$$\tan 2\theta = \frac{2c_2}{1 - C^2} = \frac{2(2.70)}{1 - 15.85} = \frac{5.40}{-14.85}$$

Set 5.40 on "C" to 14.85 on "D".
Opposite left index of body read 20.0° on "T<45".
Since $\tan 2\theta$ is negative,
$2\theta = 180° - 20.0° = 160.0°$, and $\theta = 80.0°$.
To convert θ to radians set 180 on "CF" to π right on "DF".
Opposite 80.0 on "CF" read $\theta = 1.395$ radians on "DF".

$$\tanh 2u = \frac{2c_1}{1 + C^2} = \frac{2(2.93)}{1 + 15.85} = \frac{5.86}{16.85}$$

Set 16.85 on "C" to 5.86 on "D"
Opposite left index of slide read $2u = 0.363$ on "Th".
Hence $u = 0.181$.
$u + j\theta = \tanh^{-1}(3.98\angle 42.6°)$ = **0.181 + j1.395.**

If desired the complex number $u + j\theta$ may be converted to polar form in the usual manner.

If γ is negative, then θ will be negative. For instance, $\tanh^{-1}(3.98\angle{-42.6°}) = 0.181 - j1.395$.

Exercises

Find the following inverse hyperbolic functions.

1. $\sinh^{-1}(1.67\angle 80.2°)$
2. $\cosh^{-1}(0.800\angle 10.82°)$
3. $\tanh^{-1}(1.03\angle{-1.30°})$
4. $\cosh^{-1}(1.00\angle{-39.05°})$
5. $\sinh^{-1}(0.032\angle{-59.5°})$
6. $\tanh^{-1}(0.490\angle 34.7°)$

Answers to the above exercises.

1. 1.10 + j1.36
2. 0.231 + j0.700
3. 2.02 − j1.92
4. 0.729 − j0.916
5. 0.0162 − j0.0275
6. 0.388 + j0.317

57. Summary and Applications of Hyperbolic Functions of Complex Numbers.

A. Finding values of sinh $(u + j\,\theta)$.

1. Convert θ to degrees if it is not so given.
2. If the value of θ appears on the "ST" or "T<45" scales, then let $\sinh(u = j\,\theta) = A/\underline{\propto}$, and find A and $\propto$ from the following relations.

$$\tan \propto \; = \tan\,\theta \coth u$$
$$A \; = \sinh u\left(\frac{\cos\,\theta}{\cos \propto}\right)$$

Remember that $\propto$ *is always greater than* θ.
Remember that A *is always greater than sinh u.*

a. If θ is on the "ST" scale, set θ on "ST" to u on "Th". Read $\propto$ either on "ST" scale opposite right body index or on "T<45" scale opposite left body index.

To find A, set $\propto$ on red "S" scale to u on "Sh1" or "Sh2". Opposite θ on red "S" scale, read A on "D". (If A lies off the "D" scale initially, interchange indices and read as before.) The decimal point is located from a mental estimate based on approximate values for sinh u, cos θ, and cos $\propto$.

b. If θ is on the black "T<45" scale, set θ on "T<45" to u on "Th".

(1) If slide projects to the right, read $\propto$ on black "T<45" scale opposite right index of body.

(2) If slide projects to the left, opposite left index of body, read $\propto$ on black "T>45".

Find A as explained above in part *a*.

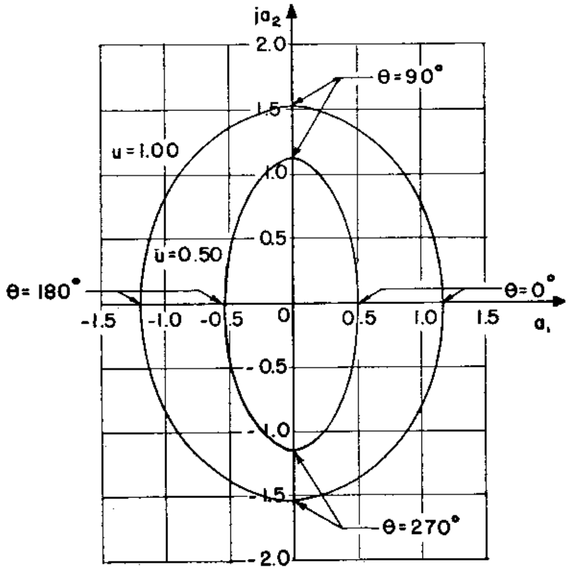
c. If θ is on the "T>45" scale, set θ on "T>45" scale. Opposite u on the "Th" scale, read $\propto$ on "T>45" scale opposite right index of body. Find A as explained above in part *a*.

3. Exceptional cases: See the explanation of special cases at the end of Section 55-A. One may always resort to the fundamental definition to find $\sinh(u + j\,\theta)$ in component form.

$$\sinh(u + j\,\theta) = \sinh u \cos\,\theta + j \cosh u \sin\,\theta$$

The value may be converted to polar form in the usual manner. (See Section 54.)

Since the hyperbolic sine of a complex number is a complex number also, values of $\sinh(u + j\,\theta)$ determine points in a complex plane. If we keep $u = 1.00$ and let θ vary from $0°$ to $360°$, the points corresponding to the varying values of $\sinh(1.00 + j\,\theta)$ determine an ellipse in the complex plane. The semi-axes of the ellipse are equal to sinh 1.00 = 1.175 and cosh 1.00 = 1.543. The point corresponding to $\theta = 0°$ lies on the positive segment of the horizontal axis, for $\sinh(1.00 + j0.0°) = \sinh 1.00 \cos 0° + j \cosh 1.00 \sin 0° = \sinh 1.00 + j0$. Similarly, the point corresponding to $90°$ lies on the positive j-axis, etc. As θ increases from $0°$ to $360°$, the point locating the value of $\sinh(1.00 + j\,\theta)$ travels counterclockwise around the ellipse, starting and ending on the positive segment of the horizontal axis.



Graphs of $\sinh(u + j\,\theta) = a_1 + ja_2$ for u = 1.00 and u = 0.50.

Fig. 74

In Figure 74 the ellipses corresponding to $\sinh(1.00 + j\,\theta)$ and $\sinh(0.50 + j\,\theta)$ have been plotted in a complex plane. Besides giving a visual description of the function $\sinh(u + j\,\theta)$, the graphs can aid in locating the decimal point in calculations.

1. If u is less than 0.50, the point corresponding to $\sinh(u + j\theta)$ will lie inside the smaller ellipse at a position which may be estimated from knowledge of the values of u and θ. Hence the magnitude (A) of $\sinh(u + j\theta)$ must be less than $\cosh 0.50 = 1.13$. If $u = 0$, then $\sinh(u + j\theta) = j \sin \theta$, and the tracing point moves up and down the j-axis between the limits +1 and −1 as θ varies from 0° to 360°.

2. If the value of u lies between 0.50 and 1.00, then the point corresponding to $\sinh(u + j\theta)$ will lie between the two ellipses at a distance (A) from the origin, where A ranges between $\sinh 0.50 = 0.521$ and $\cosh 1.00 = 1.543$.

3. If the value of u is greater than 1.00, the point corresponding to $\sinh(u + j\theta)$ will lie outside the larger ellipse and at a distance (A) from the origin, where A is greater than $\sinh 1.00 = 1.175$.

B. *Finding values of cosh $(u + j\theta)$.*

1. Convert θ to degrees if it is not so given.

2. If the value of θ appears on the "ST", "T<45" or "T>45" scales, then let $\cosh(u + j\theta) = B\underline{/\beta}$, and find B and β from the relations:

$$\tan \beta = \frac{\tan \theta}{\coth u}$$

$$B = \sinh u \left(\frac{\sin \theta}{\sin \beta}\right)$$

Remember that β *is always less than* θ.
Remember that B *is always greater than sinh u*.

a. If θ is on the "ST" scale, set θ on "ST" to the right index of the body.

(1) Opposite u on "Th" read β on "ST".

(2) If u on "Th" is beyond the left index of the slide, interchange indices and read β as one-tenth the value found on the "ST" scale opposite u on "Th".

To find B set β on black "S" scale to u on "Sh1" or "Sh2". Opposite θ on black "S" scale read B on "D". (If B lies off the "D" scale initially, interchange indices and read as before.) The decimal point is located from a mental estimate based on approximate values for sinh u, sin θ, and sin β.

b. If θ is on the black "T<45" scale, set θ on "T<45" to the right index of the body.

(1) Opposite u on "Th" read β on "T<45".

(2) If u on "Th" is beyond the left index of the slide, interchange indices and read β on "ST" opposite u on "Th".

Find β as explained above in part a.

c. If θ is on "T>45" set the index of the slide to u on "Th" and move the hairline to θ on "T>45".

(1) If the slide projects to the right, close the rule and read β under the hairline on "T<45".

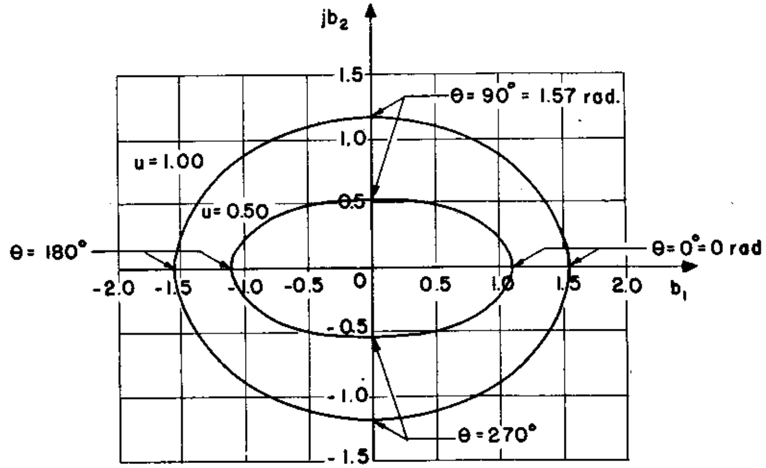
(2) If the slide projects to the left, close the rule and read β under the hairline on "T>45".

Find B as explained above in part a.

3. Exceptional cases: See the explanation of special cases at the end of Section 55-B. One may always resort to the fundamental definition to find $\cosh(u + j\theta)$ in component form.

$$\cosh(u + j\theta) = \cosh u \cos \theta + j \sinh u \sin \theta$$

The value may be converted to polar form in the usual manner. (See Section 54.)



Graphs of $\cosh(u + j\theta) = b_1 + jb_2$ for $u = 1.00$ and $u = 0.50$.

Fig. 75

In Figure 75 the ellipses corresponding to $\cosh(1.00 + j\,\theta)$ and $\cosh(0.50 + j\,\theta)$ have been plotted in a complex plane. The values of $\cosh(u + j\,\theta)$ for other values of u can be located approximately on this graph as a check on calculations and as an aid in locating decimal points.

1. If u is less than 0.50, the point corresponding to $\cosh(u + j\,\theta)$ will lie inside the smaller ellipse. Hence the magnitude (B) of $\cosh(u + j\,\theta)$ will be less than $\cosh 0.50 = 1.13$. If $u = 0$, then $\cosh(u + j\,\theta) = \cos\theta$, and the tracing point moves back and forth along the horizontal axis between the limits $+1$ and -1 as θ varies from 0° to 360°.

2. If the value of u lies between 0.50 and 1.00, then the point corresponding to $\cosh(u + j\,\theta)$ will lie between the two ellipses at a distance (B) from the origin where B ranges between $\sinh 0.50 = 0.521$ and $\cosh 1.00 = 1.543$.

3. If the value of u is greater than 1.00, the point corresponding to $\cosh(u + j\,\theta)$ will lie outside the larger ellipse and at a distance (B) from the origin where B is greater than $\sinh 1.00 = 1.175$.

C. *Finding the value of tanh $(u + j\,\theta)$.*

1. Convert θ to degrees if it is not so given.

2. Let $\tanh(u + j\,\theta) = C\underline{/\gamma}$, and find C and γ from the following relations.

$$\tan\gamma = \frac{\sin 2\theta}{\sinh 2u}$$

$$C = \sqrt{\frac{\cosh 2u - \cos 2\theta}{\cosh 2u + \cos 2\theta}}$$

a. Write down the values for $2u$ and 2θ.

b. Find the angle γ.

(1) If the value of $2u$ lies on the "Sh1" or "Sh2" scale, set 2θ on the black "S" scale to $2u$ on "Sh1" or "Sh2". Opposite either the right or left index of the body read the value of $\tan\gamma$ on "C". (If 2θ is greater than 90°, express it as $2\theta = 180° \pm \phi$, and set ϕ on black "S" scale to $2u$.) Determine the decimal point in the value of $\tan\gamma$ by a mental estimate of values of $\sin 2\theta$ and $\sin 2u$, and give the value of $\tan\gamma$ its proper sign as determined by the sign of $\sin 2\theta$. If the magnitude of $\tan\gamma$ is less than 1.00, set hairline to $\tan\gamma$ on "C" and read γ on the "ST" or black "T<45" scales under the hairline. If the magnitude of $\tan\gamma$ is greater than 1.00, set hairline to $\tan\gamma$ on "C" and read γ on the "T>45" scale under the hairline. Give γ the proper sign corresponding to the sign of $\tan\gamma$.

(2) If the value of $2u$ does not lie on the "Sh1" or "Sh2" scale, determine the value of $\sinh 2u$ by methods given in Section 53. Then set 2θ (or ϕ) on the black "S" scale to $\sinh 2u$ on "D" and read the value of $\tan\gamma$ on "C" opposite either body index. Proceed as in (1) above.

c. Determine the value of C.

(1) Find $\cosh 2u$ by methods given in Section 53.

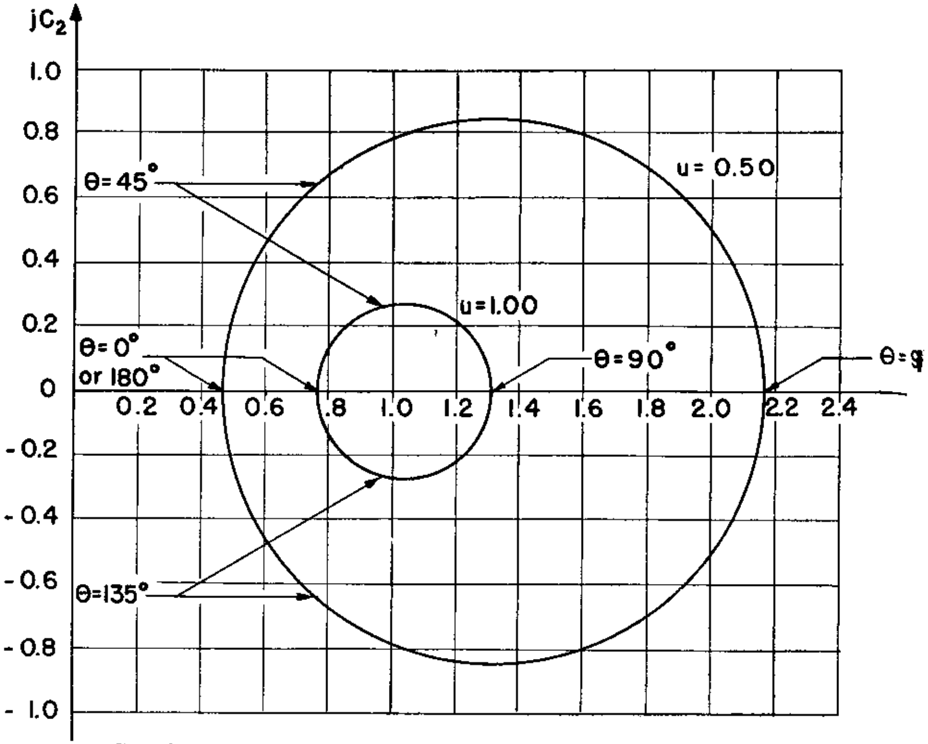
(2) Find $\cos 2\theta$ by methods given in Section 28. If 2θ is greater than 90°, let $2\theta = 180° \pm \phi$ and find $\cos 2\theta = -\cos\phi$ by the above methods.

(3) Calculate $C = \sqrt{\frac{\cosh 2u - \cos 2\theta}{\cosh 2u + \cos 2\theta}}$

Add and subtract the values found in (1) and (2) above, with due regard for the sign of $\cos 2\theta$, and obtain the numberator n and denominator d of the fraction under the radical. Then set d on the "B" scale to n on the "A" scale, being sure that the correct halves of these scales are used. Opposite either index of the slide read the value of C on the "D" scale.

3. Exceptional cases: If u and θ are less than 0.2, the accuracy of calculating C is low because cosh u and $\cos\theta$ are nearly equal. In this case and in cases for which u or θ do not appear on the rule, one should use the fundamental equation

$$\tanh(u + j\,\theta) = \frac{\sinh 2u + j \sin 2\theta}{\cosh 2u + \cos 2\theta}$$



Graphs of $\tanh(u + j\theta) = c_1 + jc_2$ for $u = 1.00$ and $u = 0.50$.

Fig. 76

In Figure 76 are shown the circles corresponding to $\tanh(1.00 + j\theta)$ and $\tanh(0.50 + j\theta)$. The values of $\tanh(u + j\theta)$ for other values of u can be located approximately on this graph as a check on calculations and as an aid in locating decimal points.

1. If u is less than 0.50, the point corresponding to $\tanh(u + j\theta)$ will lie outside the larger circle. If $u = 0$, then the tracing point moves up the j-axis from 0 for $\theta = 0°$ to 1.00 for $\theta = 45°$, and to ∞ for $\theta = 90°$. If θ varies from 0° to −90°, the point moves correspondingly along the negative j-axis.

2. If the value of u lies between 0.50 and 1.00, then the point corresponding to the complex value of $\tanh(u + j\theta)$ will lie between the two circles at a distance C from the origin, where C ranges between $\sqrt{\dfrac{\cosh 1.00 - 1.00}{\cosh 1.00 + 1.00}} = 0.462$ and $\sqrt{\dfrac{\cosh 1.00 + 1.00}{\cosh 1.00 - 1.00}} = 2.16$.

3. If the value of u is greater than 1.00, the point corresponding to $\tanh(u + j\theta)$ will lie inside the smaller circle at a distance C between 0.735 and 1.363. As u becomes very large $\tanh(u + j\theta)$ approaches the value $1.00\underline{/0°}$ whatever may be the value of θ, i.e., the circles shrink to the point $1.00\underline{/0°}$ as u increases indefinitely.

D. *Finding inverse hyperbolic functions.*

1. Given $A\underline{/\alpha}$, find $\sinh^{-1}(A\underline{/\alpha})$.
   Convert $A\underline{/\alpha}$ to component form $a_1 + ja_2$.
   Let $\sinh^{-1}(A\underline{/\alpha}) = u + j\theta$.
   Find u and θ from the following relations.

$$\sin\theta = \frac{\dfrac{2a_2}{a_1}}{\sqrt{1+\left(\dfrac{1+a_2}{a_1}\right)^2}+\sqrt{1+\left(\dfrac{1-a_2}{a_1}\right)^2}}$$

$$\sinh u = \frac{a_1}{\cos\theta}$$

2. Given $B\underline{/\beta}$, find $\cosh^{-1}(B\underline{/\beta})$.
   Convert $B\underline{/\beta}$ to component form $b_1 + jb_2$.
   Let $\cosh^{-1}(B\underline{/\beta}) = u + j\theta$.
   Find u and θ from the following relations.

$$\cos\theta = \frac{\dfrac{2b_1}{b_2}}{\sqrt{1+\left(\dfrac{1+b_1}{b_2}\right)^2}+\sqrt{1+\left(\dfrac{1-b_1}{b_2}\right)^2}}$$

$$\sinh u = \frac{b_2}{\sin\theta}$$

3. Given $C\underline{/\gamma}$, find $\tanh^{-1} C\underline{/\gamma}$.
   Convert $C\underline{/\gamma}$ to component form $c_1 + jc_2$.
   Let $\tanh^{-1}(C\underline{/\gamma}) = u + j\theta$.
   Find u and θ from the following relations.

$$\tan 2\theta = \frac{2c_2}{1-C^2}$$

$$\tanh 2u = \frac{2c_1}{1+C^2}$$

Other inverse hyperbolic functions may be calculated by the above methods.

$$\coth^{-1}(N\angle\delta) = \tanh^{-1}\left(\frac{1}{N\angle\delta}\right) = \tanh^{-1}\left(\frac{1}{N}\angle-\delta\right).$$

$$\operatorname{sech}^{-1}(N\angle\delta) = \cosh^{-1}\left(\frac{1}{N\angle\delta}\right) = \cosh^{-1}\left(\frac{1}{N}\angle-\delta\right).$$

$$\operatorname{csch}^{-1}(N\angle\delta) = \sinh^{-1}\left(\frac{1}{N\angle\delta}\right) = \sinh^{-1}\left(\frac{1}{N}\angle-\delta\right).$$

E. *Applications of hyperbolic functions of complex numbers.*

Several problems involving engineering applications of hyperbolic functions of complex numbers have been included in Section 55. A few more are included here to show other types of problems encountered and to illustrate methods of attack in solving them.

Example 1. A symmetrical T network has series impedance $Z_1 = 1270\angle 61.7^\circ$ ohms and shunt impedance $Z_2 = 4100\angle 87.14^\circ$. Calculate the propagation constant *P* for the network. $P = 2\sinh^{-1}\frac{1}{2}\sqrt{\frac{Z_1}{Z_2}}$.

Solution: Find $\frac{1}{2}\sqrt{\frac{Z_1}{Z_2}} = A\angle\alpha = a_1 + ja_2$ in component form.

Set 4100 on B-RIGHT to 1270 on A-RIGHT.
Opposite 0.5 on "C" read A = 0.278 on "D".
$\alpha = \frac{1}{2}(61.7^\circ - 87.14^\circ) = \frac{1}{2}(-25.44^\circ) = -12.72^\circ.$
Set right index of slide to 0.278 on "D"
Opposite red 12.72° on "S" read $a_1 = 0.271$ *on "D".*
Set left index of slide to 0.278 on "D".
Opposite black 12.72° on "S" read $a_2 = -0.0612$ *on "D"*
Let $\sinh^{-1}(a_1 + ja_2) = u + j\theta$.

$$\sin\theta = \frac{\dfrac{2a_2}{a_1}}{\sqrt{1+\left(\dfrac{1+a_2}{a_1}\right)^2} + \sqrt{1+\left(\dfrac{1-a_2}{a_1}\right)^2}}$$

Set 0.271 on "C" to 0.9388 on "D".
Opposite left index of slide read
$\left(\frac{1+a_2}{a_1}\right)^2 = 10.9$ *on "A".*
Set hairline to 11.9 on A-RIGHT.

Under hairline on "D" read $\sqrt{1+\left(\frac{1+a_2}{a_1}\right)^2} = 3.45.$
Set 0.271 on "C" to 1.0612 on "D".
Opposite right index of slide read
$\left(\frac{1-a_2}{a_1}\right)^2 = 15.35$ *on "A".*
Set hairline to 16.35 on A-RIGHT.
Under hairline on "D" read $\sqrt{1+\left(\frac{1-a_2}{a_1}\right)^2} = 4.05.$

$$\sin\theta = \frac{2(-0.0612)}{0.271(3.45+4.05)} = \frac{-0.1224}{0.271(7.50)}$$

Set 0.271 on "C" to 0.1224 on "D".
Opposite 7.50 on "CIF" read sin θ = 0.0603 on "DF".
Set hairline to 0.0603 on "C" and read θ = – 3.46° on "ST".

$$\sinh u = \frac{a_1}{\cos\theta}$$

Set red 3.46° on "S" to 0.271 on "D".
Opposite right index of slide read u = 0.268 on "Sh1".

$$P = 2\sinh^{-1}\tfrac{1}{2}\sqrt{\frac{Z_1}{Z_2}} = 2(0.268 - j3.46^\circ)$$
$$= 0.536 - j6.92^\circ.$$

Set 180 on "CF" to π right on "DF".
Opposite 6.92 on "CF" read θ = 0.121 radians on "DF".
P = **0.536 – j0.121.**

Example 2. A telephone line has a propagation constant $P = 0.0844\angle 51.4^\circ$ per mile at a frequency of 800 cycles per second. If the line is open-circuited at a distance x = 50 miles from the sending end, find the ratio of the open-circuit voltage V_o to the sending end voltage V_s.

$$\frac{V_o}{V_s} = \frac{1}{\cosh(Px)}$$

Solution: Express Px as u + j θ.
$Px = 50(0.0844\angle 51.4^\circ) = 4.22\angle 51.4^\circ.$
Set right index of slide to 4.22 on "D".

Opposite 51.4° on red "S" scale read u = 2.63 on "D".
Opposite 51.4° on black "S" scale read θ *= 3.295 on "D".*
Set 180 on "CF" to π *right on "DF".*
Opposite 3.295 on "D" read $\theta = 188.8°$.
$Px = u + j\theta = 2.63 + j188.8°$.
$\text{Cosh}(u + j\theta) = B\underline{/\beta}$.

$$\text{Tan}\ \beta = \frac{\tan\ \theta}{\coth\ u} = \frac{\tan 8.8°}{\coth 2.63}$$

Set 8.8° on "T<45" to right index of body.
Opposite 2.63 on "Th" read β *= 8.7° on "T<45".*

$$B = \sinh u \left(\frac{\sin\ \theta}{\sin\ \beta}\right) = \sinh 2.63\ \frac{\sin(-8.8°)}{\sin(8.7°)}$$

Set black 8.7° on "S" to 2.63 on "Sh2".
Opposite 8.8° on "S" read B = −6.99 on "D".
$\cosh(Px) = -6.99\underline{/8.7°}$.

$$\frac{V_o}{V_s} = \frac{1}{\cosh(Px)} = \frac{1}{-6.99\underline{/8.7°}}$$

Set hairline to 6.99 on "C".
Under hairline on "CI" read 0.143.

$$\frac{V_o}{V_s} = -0.143\underline{/-8.7°} = \mathbf{0.143\underline{/171.3°}}.$$

Example 3. A tube of length L = 1000 centimeters has one end open and the other end closed by a driving piston. The specific acoustic impedance z_o offered to the driving piston is given by
$z_o = 42 \tanh[\pi(0.0484 + 0.000363L)$
$-j\pi(0.145 + 0.2\ L)]$. Find z_o.

Solution: $z_o = 42 \tanh(u + j\theta)$.
$u = \pi(0.0484 + 0.363) = \pi(0.4114)$.
Set hairline to 0.4114 on "D".
Under hairline on "DF" read u = 1.282
$\theta = \pi(0.145 + 200) = \pi(200.145)$.
After subtracting 200π *radians we have*
$\theta = \pi(0.145)$ *radians.*

Set left index of slide to 180 on "D".
Opposite 0.145 on "C" read $\theta = 26.1°$ *on "D".*
$\tanh(u + j\theta) = \tanh(1.282 + j26.1°) = C\underline{/\gamma}$.

$$\tan\ \gamma = \frac{\sin 2\theta}{\sinh 2u} = \frac{\sin 52.2°}{\sinh 2.564}.$$

Set black 52.2° on "S" to 2.564 on "Sh2".
Opposite left index of body read γ *= 6.97° on "T<45".*

$$C = \sqrt{\frac{\cosh 2u - \cos 2\theta}{\cosh 2u + \cos 2\theta}}$$

$$= \sqrt{\frac{\cosh 2.564 - \cos 52.2°}{\cosh 2.564 + \cos 52.2°}}$$

Set right index of slide to 2.564 on "Th".
Opposite 2.564 on "Sh2" read cosh 2.564 = 6.54 on "C".
Set hairline to red 52.2° on "S".
Under hairline on "C" read cos 52.2° = 0.613.

$$C = \sqrt{\frac{6.54 - 0.613}{6.54 + 0.613}} = \sqrt{\frac{5.927}{7.153}}$$

Set 7.153 on B-LEFT to 5.927 on A-LEFT.
Opposite right index of slide read C = 0.912 on "D".
$z_o = 42 \tanh(u + j\theta) = 42(0.912\underline{/6.97°})$ mech. ohms.
Set right index of slide to 42 on "D".
Opposite 0.912 on "C" read 38.25 on "D".
$z_o = \mathbf{38.2\underline{/6.97°}}$ *mechanical ohms.*

Exercises

1. A telephone line 25 miles long has a propagation constant $P = 0.0844\underline{/51.4°}$ per mile. The voltage at the receiving end is to be $V_R = 1.00$ volt. What voltage V_s must be impressed on the sending end of the line? $V_s = V_R(\cosh PL + \sinh PL)$, where *L* is the length of the line in miles and where the load impedance has been matched to the line.

2. A coaxial wave guide has inner and outer radii of a = 0.5 centimeters and b = 2.0 centimeters respectively. Its characteristic impedance Z_o is approximately a pure resistance equal to $Z_o = 138 \log\left(\frac{b}{a}\right)$. For a wave guide open at one end the im-

pedance Z at a point x centimeters back along the tube is given by $Z = Z_0 \coth \gamma x$, where γ is the propagation constant per centimeter. Taking $\gamma = 0.013 + j0.126$ at a frequency of 600 megacycles, calculate Z at a distance $x = 40$ cm. back from the open end of the tube.

3. The acoustic impedance at the end of a hollow tube is given by $Z_L = 42\ S \tanh \pi (\alpha_L + j\beta_L)$ mechanical ohms, where S is the cross-sectional area of the tube in square centimeters. Z_L is measured to be $120 + j14.5$ mechanical ohms. Calculate α_L and β_L, if the tube radius is 1.00 centimeter.

4. A section of a loaded open wire telephone line is represented by a symmetrical T network having series impedance $Z_1 = 1{,}370\underline{/88.3^\circ}$ ohms and shunt impedance $Z_2 = 2{,}800\underline{/-89.1^\circ}$ ohms. Find the propagation constant $P = \cosh^{-1}\left[1 + \frac{Z_1}{2Z_2}\right]$.

Answers to the above exercises.

1. $V_s = 3.73\underline{/94.46^\circ}$ volts.
2. $Z = 48.2\underline{/-26.2^\circ}$ ohms.
3. $\alpha_L = 0.47$ and $\beta_L = 0.144$.
4. $P = 0.0169 - j0.715$.

CHAPTER X
MATHEMATICAL FORMULAE

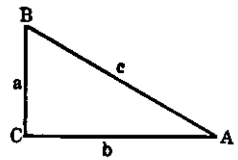
Plane Trigonometry.

Right Triangle

$\sin A = \frac{a}{c}$ $\qquad \cos A = \frac{b}{c}$

$\tan A = \frac{a}{b}$ $\qquad \cot A = \frac{b}{a}$

$\sec A = \frac{c}{b}$ $\qquad \operatorname{cosec} A = \frac{c}{a}$



$$\sin A = \cos\left(\frac{\pi}{2} - A\right) = -\cos\left(\frac{\pi}{2} + A\right)$$

$$\cos A = \sin\left(\frac{\pi}{2} - A\right) = \sin\left(\frac{\pi}{2} + A\right)$$

$$\tan A = \cot\left(\frac{\pi}{2} - A\right) = -\cot\left(\frac{\pi}{2} + A\right)$$

$$\cot A = \tan\left(\frac{\pi}{2} - A\right) = -\tan\left(\frac{\pi}{2} + A\right)$$

$$\sec A = \operatorname{cosec}\left(\frac{\pi}{2} - A\right) = \operatorname{cosec}\left(\frac{\pi}{2} + A\right)$$

$$\operatorname{cosec} A = \sec\left(\frac{\pi}{2} - A\right) = -\sec\left(\frac{\pi}{2} + A\right)$$

$\sin(-A) = -\sin A$ $\qquad \cos(-A) = \cos A$

$\tan(-A) = -\tan A$ $\qquad \cot(-A) = -\cot A$

$\sec(-A) = \sec A$ $\qquad \operatorname{cosec}(-A) = -\operatorname{cosec} A$

NUMERICAL VALUES

Angle	0°	30°	45°	60°	90°
sin	0	½	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	½	0
tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	∞
cot	∞	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

MATHEMATICAL FORMULAE

Plane Geometrical Figures

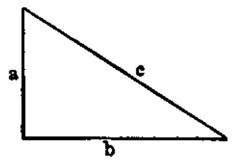
Right Triangle

$c = \sqrt{a^2 + b^2}$

$a = \sqrt{c^2 - b^2}$

$b = \sqrt{c^2 - a^2}$

area $= \frac{1}{2}\,ab$

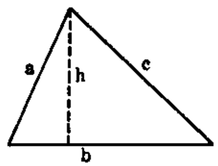


Any Triangle

area $= \frac{1}{2}\,bh$

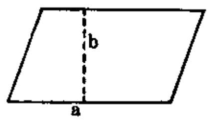
area $= \sqrt{s(s-a)\ (s-b)\ (s-c)}$

$s = \frac{1}{2}\ (a+b+c)$



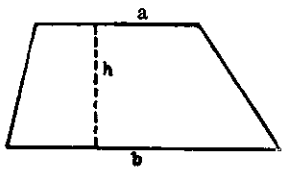
Parallelogram

area $= ab$



Trapezoid

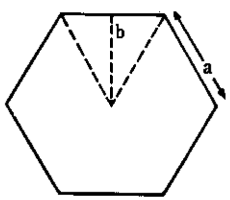
area $= \frac{1}{2}h\ (a+b)$



Regular Polygon

area $= \frac{1}{2}\,abn$

n = number of sides



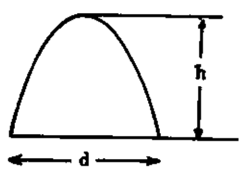
Parabola

$$\text{length of arc} = \frac{d^2}{8\,h}\left[\sqrt{c\ (1+c)} + 2.0326\ \log_{10}(\sqrt{c} + \sqrt{1+c})\right]$$

in which

$$c = \left(\frac{4\,h}{d}\right)^2$$

area $= \frac{2}{3}\,dh$



MATHEMATICAL FORMULAE

Plane Geometrical Figures.

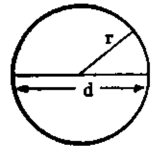
Circle

circumference $= 2\ \pi r$

$= \pi d$

area $= \pi r^2$

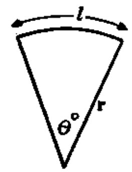
$= \pi \dfrac{d^2}{4}$



Sector of Circle

arc $= l = \pi r \dfrac{\theta^\circ}{180^\circ}$

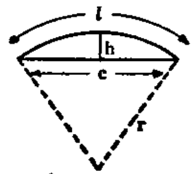
area $= \frac{1}{2}\,rl = \pi r^2 \dfrac{\theta^\circ}{360^\circ}$



Segment of Circle

chord $= c = 2\sqrt{2\ hr - h^2}$

area $= \frac{1}{2}\,rl - \frac{1}{2}\,c\ (r-h)$



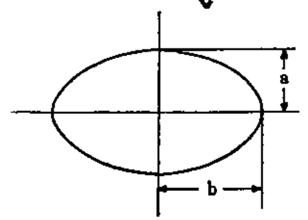
Ellipse

circumference =

$$\pi\ (a+b)\frac{64 - 3\left(\dfrac{b-a}{b+a}\right)^4}{64 - 16\left(\dfrac{b-a}{b+a}\right)^2}$$

(close approximation)

area $= \pi\ ab$

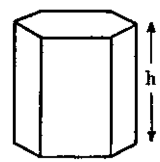


Solid Geometrical Figures.

Right Prism

lateral surface = perimeter of base $\times\ h$

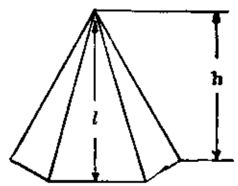
volume = area of base $\times\ h$



Pyramid

lateral area $= \frac{1}{2}$ perimeter of base $\times\ l$

volume = area of base $\times \dfrac{h}{3}$



MATHEMATICAL FORMULAE

Solid Geometrical Figures.

Frustum of Pyramid

lateral surface $= \frac{1}{2}\, l\, (P + p)$

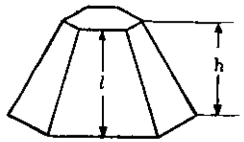
P = perimeter of lower base

p = perimeter of upper base

volume $= \frac{1}{3}h\, [A + a + \sqrt{Aa}]$

A = area of lower base

a = area of upper base

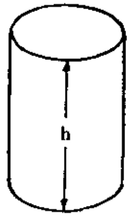


Right Circular Cylinder

lateral surface $= 2\, \pi r h$

r = radius of base

volume $= \pi r^2 h$

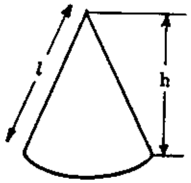


Right Circular Cone

lateral surface $= \pi r l$

r = radius of base

volume $= \frac{1}{3}\, \pi r^2 h$



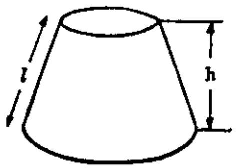
Frustum of Right Circular Cone

lateral surface $= \pi l\, (R + r)$

R = radius of lower base

r = radius of upper base

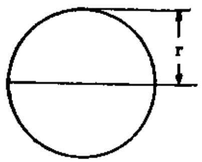
volume $= \frac{1}{3}\, \pi h\, [R^2 + Rr + r^2]$



Sphere

surface $= 4\, \pi r^2$

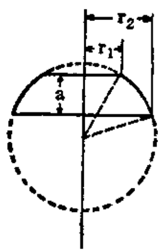
volume $= \dfrac{4}{3}\, \pi r^3$



Segment of Sphere

volume of segment

$= \dfrac{1}{6}\, a\, \pi\, [3\, (r_1{}^2 + r_2{}^2) + a^2]$



MATHEMATICAL FORMULAE

Spherical Trigonometry.

Right Spherical Triangles

$\cos c = \cos a \cos b$

$\sin a = \sin c \sin A$

$\sin b = \sin c \sin B$

$\cos A = \cos a \sin B$

$\cos B = \cos b \sin A$

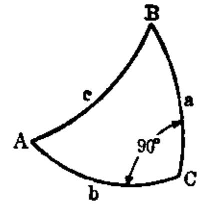
$\cos A = \tan b \cot c$

$\cos B = \tan a \cot c$

$\sin b = \tan a \cot A$

$\sin a = \tan b \cot B$

$\cos c = \cot A \cot B$



Oblique Spherical Triangles

$\dfrac{\sin a}{\sin A} = \dfrac{\sin b}{\sin B} = \dfrac{\sin c}{\sin C}$

$\cos a = \cos b \cos c + \sin b \sin c \cos A$

$\cos A = \sin B \sin C \cos a - \cos B \cos C$

$\cot a \sin b = \cot A \sin C + \cos C \cos b$

$s = \frac{1}{2}\, (a + b + c)$

$S = \frac{1}{2}\, (A + B + C)$

$\sin\left(\dfrac{A}{2}\right) = \sqrt{\dfrac{\sin\,(s-b)\sin\,(s-c)}{\sin b \sin c}}$

$\cos\left(\dfrac{A}{2}\right) = \sqrt{\dfrac{\sin s \sin\,(s-a)}{\sin b \sin c}}$

$\tan\left(\dfrac{A}{2}\right) = \sqrt{\dfrac{\sin\,(s-b)\sin\,(s-c)}{\sin s \sin\,(s-a)}}$

$\sin\left(\dfrac{a}{2}\right) = \sqrt{-\dfrac{\cos S \cos\,(S-A)}{\sin B \sin C}}$

$\cos\left(\dfrac{a}{2}\right) = \sqrt{\dfrac{\cos\,(S-B)\cos\,(S-C)}{\sin B \sin C}}$

$\tan\left(\dfrac{a}{2}\right) = \sqrt{-\dfrac{\cos S \cos\,(S-A)}{\cos\,(S-B)\cos\,(S-C)}}$

$\tan \frac{1}{2}\,(a-b) = \dfrac{\sin \frac{1}{2}\,(A-B)}{\sin \frac{1}{2}\,(A+B)} \tan \frac{1}{2} c$

$\tan \frac{1}{2}\,(a+b) = \dfrac{\cos \frac{1}{2}\,(A-B)}{\cos \frac{1}{2}\,(A+B)} \tan \frac{1}{2} c$

$\tan \frac{1}{2}\,(A-B) = \dfrac{\sin \frac{1}{2}\,(a-b)}{\sin \frac{1}{2}\,(a+b)} \cot \frac{1}{2} C$

$\tan \frac{1}{2}\,(A+B) = \dfrac{\cos \frac{1}{2}\,(a-b)}{\cos \frac{1}{2}\,(a+b)} \cot \frac{1}{2} C$

$\tan \frac{1}{2} c = \dfrac{\sin \frac{1}{2}\,(A+B) \tan \frac{1}{2}\,(a-b)}{\sin \frac{1}{2}\,(A-B)}$